I. Overview

Birational geometry: study/classify algebraic varieties (reduced & irreducible) $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ up to birational equivalence.

X and Y are birational if $\exists U \subseteq X$, $V \subseteq Y$ open dense s.t. $U \cong V$ (isomorphism).

Equivalently: $K(X) = K(Y)$ where $K(-)$ denotes the function field.

Mori theory / Mori program / Minimal model program

M1 Find a good rep in each birat. class

M2 Study good reps

M3 Understand the relation b/w two good reps in the same class

Thm. (Hironaka, '64) X variety, then $\exists \pi : X' \to X$ birational morphism s.t. X' smooth.

When X is a smooth variety: take the canonical bundle $\omega_X = \Lambda^n \Omega^1_X = K_X$, always an invertible sheaf. $(n = \dim X)$

For curves: two smooth (projective) birational curves are isomorphic b/c smooth proj curves are determined by their function field.

C	$g(C) = h^0(C, \omega_C)$	topology
$\mathbb{P}^1$	0	sphere
elliptic	1	torus
of general type	≥ 2	

Kodaira dimension for $\dim X = n$, $R(X, \omega_X) = \bigoplus_{m \geq 0} H^m(X, \omega_X^{\otimes m})$ canonical ring,

then $k(X) := \begin{cases} -\infty & \text{if } h^0(X, \omega_X^{\otimes m}) = 0 \quad \forall m > 0 \\ \operatorname{tr\,deg} R(X, \omega_X) & \text{otw.} \end{cases}$

This is a birat. invariant, and we have $k(X) = -\infty$ or $0 \leq k(X) \leq \dim X$

Surfaces: S smooth surface, $p \in S$, $\pi : S' \to S$ blow-up at p,

$(S' \setminus E) \xrightarrow[\text{iso}]{\cong} (S \setminus p)$ where $E \cong \mathbb{P}^1$.

We can blow up as many points as we want.

M1 (Castelnuovo, ~1900) S surface, $C \subseteq S$ curve s.t. $\overbrace{C \cong \mathbb{P}^1 \text{ and } \omega_S C = -1}^{C \text{ is called a } (-1)\text{-curve}}$, where $\omega_S C = \deg(\omega_S|_C)$,

then S is the blowup of some smooth surface S_1,

and C is the exceptional locus.

MMP: Repeat until you don't have any (-1)-curve, get $S \to S_1 \to \dots \to S_n = T$ s.t. T has no (-1)-curve.

M2: two cases.

- If $k(S) = -\infty$ then T is either $\mathbb{P}^1$ or a ruled surface, i.e. $T \xrightarrow{\pi} B$ where B is a curve and the fibres of π are $\mathbb{P}^1$.
- If $k(S) \geq 0$ then T is called a minimal model, and ω is a nef, i.e. $\omega_T C \geq 0 \quad \forall C \subseteq T$ curve.

Moreover, for $m \gg 0$, $\omega_T^{\otimes m}$ is globally generated, it gives a morphism

$$\varphi_{|\omega_T^{\otimes m}|} : T \longrightarrow B = \operatorname{Proj} \bigoplus_{l \geq 0} H^0(T, \omega_T^{\otimes l}) = \operatorname{Proj} \bigoplus_{l \geq 0} H^0(S, \omega_T^{\otimes l})$$

canonical model of S $\qquad$ $\dim B = k(S) = k(T)$

$\rightsquigarrow$ Enriques classification of surfaces

$k(S)$	birational examples
$-\infty$	$\mathbb{P}^2$, $\mathbb{P}^1 \times C$ (C curve)
0	$K3$ surfaces, Abelian surfaces
1	elliptic surfaces
2	surfaces of general type

M3: • Minimal models are unique

• Ruled surfaces are connected by elementary transformations.

Higher dimension: a variety Y (with appropriate singularities) is a minimal model if ω_Y is a nef. (This is a definition.)

M1: Conj.: Given a smooth var. X there is a MMP for X, i.e. a sequence of birat. maps $X \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \dots \dashrightarrow X_m = Y$ (not nec. def'd everywhere) s.t. either Y is a minimal model ($\Leftrightarrow k(X) \geq 0$)

or Y is a Mori fibre space, i.e. $\exists \pi : Y \to B$ s.t. $\dim B < \dim Y$, and the fibres F of π are Fano varieties (i.e. $\omega_F^{-1} - K_F$ ample).

dim 3, 4 : proven (Mori, Shokurov, Kollar, Kawamata, Reid et al., '80)

Birkar – Cascini – Hacon – McKernan 2010 for var. of gen. type, i.e. $k(X) = \dim X$

Cascini – Lazic 2012

Mori proved the first versions of the Cone and Contraction thms, using bend and break.

M2: Conjecture (Abundance conj.): Y is a minimal $\Rightarrow$ $\omega_Y^{\otimes m}$ glob. gen'd for $m \gg 0$.

$\Rightarrow$ $\varphi_{|\omega_Y^{\otimes m}|} : Y \to B$ where $\operatorname{Proj} R(Y, \omega_Y) = B$.

True when $k(Y) = \dim Y$, Kawamata – Shokurov

Two cases:

- $k(Y) = \dim Y$ $\Rightarrow$ φ birational, K_B ample
- $0 \leq k(Y) \leq n-1$ $\Rightarrow$ the fibres F of φ are Calabi–Yau varieties, i.e. $\omega_F \sim 0$ $(k(F)=0)$

M3: Thm (Kawamata) Two birational minimal models are connected by so-called flops.

Thm. (Hacon – McKernan) Two birational Mori fibre spaces are connected by so-called Sarkisov links.

Birkar (Fields Medal): Boundedness of Fano varieties, MMP

II. Divisors and intersection numbers

Divisors: q.v. Hartshorne II§6.

Intersection: q.v. Debarre 1.2

II.1. Weil divisors

X normal variety over k

Def. A prime divisor Z of X is a (reduced, irreducible, closed) subvariety of X of codim 1.

The group of Weil divisors is the free ab. gp. $\mathrm{WDiv}(X)$ generated by prime divisors.

That is, a Weil divisor D is of the form $\sum d_n Z_n$, Z_n prime div., $d_n \in \mathbb{Z}$

D is called effective if $\forall d_n \geq 0$

For $f \in K^*(X)$ rational function, the divisor of f is $(f) := \sum_{Z \text{ prime div}} \operatorname{ord}_Z(f) Z$

Can be shown: only finitely many terms.

If $\eta \in Z$ is the generic point, $\mathcal{O}_{X,\eta}$ is a DVR by normality, ord_Z is the valuation on $K^*(X)$

Def. $D \in \mathrm{WDiv}(X)$ is principal if $\exists f \in K^\times(X)$ s.t. $D = (f)$.

D, D' are linearly equivalent if $D - D'$ is principal.

The quotient of WDiv under lin. equivalence is $\mathrm{Cl}(X)$, the class group.

Ex. $\mathrm{Cl}\,\mathbb{P}^n = \mathbb{Z}$, generated by a hyperplane.

$Q \subseteq \mathbb{P}^3$ quadric, smooth $\Rightarrow$ $\mathrm{Cl}\,Q = \mathbb{Z} \oplus \mathbb{Z}$ since $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$

Problems: Weil divisors don't play nice with pullbacks and intersections.

II.2. Cartier divisors

X scheme, $\mathcal{M}_X$ the sheaf of total quotient rings of $\mathcal{O}_X$, i.e. the unique sheaf s.t. if $U = \mathrm{Spec}\, A \subseteq X$ open affine subset then $\Gamma(U, \mathcal{M}_X) = A_{tot}$ total ring of fractions, the localisation of A at the set of non-zero divisors.

Rmk. If X is integral $\Rightarrow$ $\mathcal{M}_X = K(X)$ is the constant sheaf of rational functions

Def. The group of Cartier divisors is $\mathrm{Div}(X) := \Gamma\left(X, \mathcal{M}_X^\times / \mathcal{O}_X^\times\right)$

A Cartier divisor is rep'd by $\{(U_i, f_i)\}$ where $\{U_i\}$ is an open cover of X and $f_i \in \Gamma(U_i, \mathcal{M}_X^\times)$ s.t. $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X^\times)$

f_i is called a local equation of D.

$D \in \mathrm{Div}(X)$ is principal if it is in the img of $\Gamma(X, \mathcal{M}_X^\times) \longrightarrow \Gamma(X, \mathcal{M}_X^\times / \mathcal{O}_X^\times)$

Prop. X normal variety $\Rightarrow$ there is an injective homomorphism of groups

$\mathrm{Div}(X) \hookrightarrow \mathrm{WDiv}(X)$.

Pf (SKETCH): $D \in \mathrm{Div}(X)$ given by $\{(U_i, f_i)\}$.

Then let $$Y := \sum_{\substack{Z \text{ p.div} \\ U_i \cap Z \neq \emptyset}} \underbrace{\mathrm{ord}_Z(f_i)}_{\text{vanishing order}} Z \in \mathrm{WDiv}(X)$$

Ex. 1) X affine quadratic cone, $\underbrace{X}_{\text{normal var.}} = \{xy - z^2 = 0\} \subseteq \mathbb{A}^3$, $W := \{x = y = 0\}$

Then W is Weil but not Cartier.

Weil: clear

$\neg$Cartier: locally around the divisor W cannot be defined by 1 equation.

$2W$ is Cartier, cut by $\{y = 0\}$

2) $X = \{xy - zw = 0\} \subseteq \mathbb{A}^4$, $W = \{x = z = 0\}$, $W' = \{y = w = 0\}$

W, W' are Weil divisors, but no positive multiple of W, W' is Cartier

$W \cap W' = \{0\}$, this is not how Cartier divisors behave, the dimension decreases only by 1 when intersecting with a Cartier divisor.

3) X the projective cone over an elliptic curve $C \subseteq \mathbb{P}^1$

$\Rightarrow \mathrm{Cl}(C) \cong \mathrm{Pic}(C)$

W a ruling over a non-torsion point $C \Rightarrow W$ Weil but no pos. multiple is Cartier

17.10.2018

Let X be a scheme, $\mathrm{Pic}\, X$ the Picard gp, i.e. the gp of iso classes of invertible sheaves (that is, line bundles).

Def. X scheme, D Cartier rep'd by $\{(U_i, f_i)\}$. Then the inv'ble sheaf $\mathcal{O}_X(D)$ associated to D is the sub-$\mathcal{O}_X$-module of $\mathcal{M}_X$ def'd by f_i^{-1} on U_i.

Prop. X scheme. Then there is a 1:1 correspondence b/w Cartier divisors and inv'ble subsheaves of $\mathcal{M}_X$, $\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2) = \mathcal{O}_X(D_1 + D_2)$

Cor. There is an injective homomorphism $\mathrm{Div}(X)/\sim \hookrightarrow \mathrm{Pic}(X)$.

Thm. If X is integral or projective, this gives an iso $\mathrm{Div}(X)/\sim \xrightarrow{\cong} \mathrm{Pic}(X)$.

Effective divisors

Def. X scheme, $D \in \mathrm{Div}(X)$. Then D is effective if rep'd by $\{(U_i, f_i)\}$ s.t. $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$, i.e. effective divisors are represented by global functions.

The subscheme $Y \subseteq X$ associated to D is the closed subscheme def'd by the ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$ locally generated by the f_i.

This notion will prove useful when we do induction on divisors.

Rmk. • If X is smooth: eff Cartier $\leftrightarrow$ eff. Weil.

• $D \in \mathrm{Div}(X)$ Cartier $\Rightarrow \mathcal{J}_Y \cong \mathcal{O}_X(-D)$, and we have a short exact sequence

$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$ (here there should be pullbacks, but this is the usual way of writing this)

II.3. Intersection numbers

Thm. $D_1, \dots, D_r$ Cartier on X proper scheme, with $r \geq \dim X$.

Then the function $(m_1, \dots, m_r) \in \mathbb{Z}^r \longmapsto \chi(X, m_1 D_1 + \dots + m_r D_r)$ is a polynomial with rational coeffs and $\deg \leq \dim X$.

Def. The intersection number $D_1 \cdots D_r$ is the coeff of $m_1 \cdots m_r$ in this polyn. This turns out to be integral.

Ex. 1) X smooth curve $\Rightarrow \chi(X, mD) = \deg D \cdot m + \chi(\mathcal{O}_X)$ by RR

2) $X \subseteq \mathbb{P}^N$, $\dim X = n$, $H := \mathcal{O}_X(1)$. Then $m \mapsto \chi(X, \mathcal{O}_X(m))$ is the Hilbert polynomial, and $\deg X$ is H^n.

Prop. a) The map $D_1, \dots, D_n \mapsto D_1 \cdots D_n$ is multilinear, symmetric, takes integral values.

b) If D_1 is effective with ass. subsch. Y then $D_1 \cdots D_n = (D_2 \cdots D_n)_Y$.

(Think of this as restricting to a hypersurface.)

For a curve C, $D_C = \deg(\mathcal{O}_C(D))$

Thm. If $D_1, \dots, D_n$ on a smooth X are effective and meet transversally then $D_1 \cdots D_n = \#\{D_1 \cap \dots \cap D_n\}$ (that is, there is no multiplicity).

locally given by the intersection of n hyperplanes

Projective formula. Let $\pi : Y \longrightarrow X$ be a surjective morph. b/w proper varieties, let $D_1, \dots, D_r \in \mathrm{Div}(X)$, $r \geq \dim Y$.

Then $\pi^*(D_1) \cdots \pi^*(D_r) = \deg(\pi) \cdot D_1 \cdots D_r$.

III. Linear systems and ample divisors

III. 1. Morphisms to $\mathbb{P}^N$

Def. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on a scheme X and $s_i \in \Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F})$ a global section of $\mathcal{F}$. We say that $\mathcal{F}$ is (globally) generated by $(s_i)_i$ if $\forall x \in X$ point the stalk $\mathcal{F}_x$ is generated by $((s_i)_x)_i$ as an $\mathcal{O}_{X,x}$-module.

Rmk. $\mathcal{F}$ is globally generated iff $ev: H^0(X, \mathcal{F}) \otimes_k \mathcal{O}_X \longrightarrow \mathcal{F}$ is surjective. where X/k

Ex. $\mathcal{O}_{\mathbb{P}^n_k}(m)$ on $\mathbb{P}^n_k$ is globally generated by $\Gamma(\mathbb{P}^n_k, \mathcal{O}(m))$ for $m \geq 0$.

Thm. X scheme over a field k.

a) If $f : X \longrightarrow \mathbb{P}^N$ is a morphism then $f^* \mathcal{O}_{\mathbb{P}^N}(1)$ is an inv'ble sheaf which is globally generated by $f^* x_0, \dots, f^* x_N$.

b) Conversely, if $\mathcal{L}$ is an inv'ble sheaf and $s_1, \dots, s_N \in \Gamma(X, \mathcal{L})$ generate $\mathcal{L}$ then $\exists! \; f : X \longrightarrow \mathbb{P}^N$ morphism s.t. $\mathcal{L} = f^* \mathcal{O}_{\mathbb{P}^N}(1)$ and $s_i = f^* x_i$.

PF (SKETCH): a) easy.

b) $X_i := \{p \in X \mid s_i \notin \mathfrak{m}_p \mathcal{L}_p\}$ form an open cover of X.

$U_i := \{x_i \neq 0\} \subseteq \mathbb{P}^N$, $U_i = \mathrm{Spec}\, k[y_0, \dots, \widehat{y_i}, \dots, y_N]$, $y_j = \frac{x_j}{x_i}$

Define $k[y_0, \dots, y_n] \longrightarrow \Gamma(X_i, \mathcal{O}_{X_i})$

$y_j \longmapsto \frac{s_j}{s_i}$

These glue together.

III. 2. Linear systems and rational maps

X nonsingular proj. variety over $k = \bar{k}$.

Prop. D_0 divisor on X.

a) $\forall s \in H^0(X, \mathcal{O}_X(D_0))$ the divisor of zeroes $(s)_0$ is an effective divisor linearly equivalent to D_0.

b) Every effective divisor linearly equivalent to D_0 is of the form $(s)_0$ where $s \in H^0(X, \mathcal{O}_X(D_0))$

c) $(s)_0 = (s')_0$ iff $\exists \lambda \in k^\times$ s.t. $s' = \lambda s$.

Def. The complete linear system $|D_0|$ is the set of eff. divisors lin. eq. to D_0.

By the Prop, $|D_0| = \mathbb{P}\left(H^0(X, \mathcal{O}_X(D_0))\right)$

Def. A linear system δ on X is a linear subspace of $|D_0|$ for some divisor D_0 on X.

Notation: $\delta = |V| = \mathbb{P}(V)$, $V \subseteq H^0(X, \mathcal{O}_X(D_0))$

$ev : V \otimes \mathcal{O}_X \to \mathcal{O}_X(D_0)$ evaluation

Def. The base ideal $b(|V|)$ is the img of $V \otimes \mathcal{O}_X(D_0)^\vee \longrightarrow \mathcal{O}_X$.

The base locus $Bs(|V|)$ is the closed subscheme given by $b(|V|)$;

$Bs(|V|)$ is the set of pts of X at which all sections of V vanish.

Let $|V|$ be a linear system, $V \subseteq H^0(X, \mathcal{O}_X(D))$.

$$\varphi = \varphi_{|V|} : X \setminus Bs(|V|) \longrightarrow \mathbb{P}(V^\vee)$$

$x \longmapsto$ hyperplane of sections of V that vanish in x

This is a well-def'd morphism.

We often write $\varphi_{|V|}(x) = [s_0(x), \dots, s_n(x)]$ where $s_0, \dots, s_n$ form a basis of V.

Def. If $Bs(|V|) = \emptyset$, $|V|$ is called base point free.

If $V = H^0(X, \mathcal{L})$ then $|V|$ is bpf iff $\mathcal{L}$ is globally generated.

Conclusion: there is a 1:1 correspondence

$$\left\{ \begin{array}{c} \text{non-degenerate rational maps} \\ X \dashrightarrow \mathbb{P}^n \\ \text{up to proj. equivalence} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{linear systems } |V| \\ \text{of dimension } n \\ \text{up to isomorphism} \end{array} \right\}$$

(fill in the details of the proof).

Ex. 1) $m \geq 1$, $H^0(\mathbb{P}^1, \mathcal{O}(m)) = \langle s^m, s^{m-1}t, \dots, t^m \rangle$

This gives an embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^m$

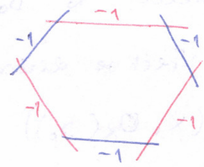
$$[s:t] \longmapsto [s^m, \dots, t^m]$$

2) Consider $\mathbb{P}^2$. $V := \langle tu, su, st \rangle \subseteq H^0(\mathbb{P}^2, \mathcal{O}(2))$

Base locus: $\{[1,0,0], [0,1,0], [0,0,1]\}$

$$\begin{array}{ccc} \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^2 \\ [s,t,u] & \longmapsto & [tu, su, st] \\ & & \left[\frac{1}{s}, \frac{1}{t}, \frac{1}{u}\right] \end{array}$$

Cremona involution



Blowup of $\mathbb{P}^2$ at $[1:0:0], [0,1,0], [0,0,1]$

III. 3. Ample divisors

Def. X proper scheme $/k$, $\mathcal{L}$ line bundle on X. $\mathcal{L}$ is called very ample if there is an embedding $\iota: X \hookrightarrow \mathbb{P}^N$ s.t. $\mathcal{L} = \iota^* \mathcal{O}_{\mathbb{P}^N}(1)$.

$\mathcal{L}$ is ample if $\mathcal{L}^{\otimes m}$ is very ample for some $m \gg 0$.

The def.s can be transferred to Cartier divisors too.

Ex. X curve. Then D is ample iff $\deg D > 0$.

Goal: numerical criteria of ampleness.

Thm. (Cartan – Serre – Grothendieck) $\mathcal{L}$ a line bundle on X. TFAE:

(1) $\mathcal{L}$ is ample.

(2) For any coherent sheaf $\mathcal{F}$ on X: $\exists m_1 > 0$ s.t. $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$ $\forall i > 0$ $\forall m \geq m_1$. Serre vanishing

(3) For any coherent sheaf $\mathcal{F}$ on X: $\exists m_2 > 0$ s.t. $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ is globally generated $\forall m \geq m_2$.

(4) $\exists m_3 > 0$ $\forall m \geq m_3$ $\mathcal{L}^{\otimes m}$ is very ample.

Thm. (Nakai – Moishezon) X a proper scheme $/k$, D Cartier on X. Then D is ample iff $\forall V \subseteq X$ integral subscheme: $D^{\dim V} V > 0$. 24.10.2018

Lemma 1. $f: X \to Y$ finite morphism b/w proper schemes, $\mathcal{L}$ an ample line bundle on $X \Rightarrow f_* \mathcal{L}$ is ample on Y.

Pf: Let $\mathcal{F}$ be a coherent sheaf on Y. We want to use Serre vanishing.

$$f_*(\mathcal{F} \otimes f^* \mathcal{L}^{\otimes m}) = f_* \mathcal{F} \otimes \mathcal{L}^{\otimes m}$$

$$H^i(Y, \mathcal{F} \otimes f^* \mathcal{L}^{\otimes m}) = H^i(X, f_* \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0 \quad \forall i > 0 \quad \forall m \gg 0 \text{ by Serre vanishing.} \quad \square$$

Mori theory, lecture 3

<u>Lemma 2.</u> X proper scheme $/k$, $\mathcal{L}$ line bundle on X.

1) $\mathcal{L}$ ample $\Leftrightarrow$ $\mathcal{L}$ restricted to X_{red} is ample.

2) $\mathcal{L}$ ample $\Leftrightarrow$ $\mathcal{L}$ restricted to the irreducible components of X is ample.

Pf: 1) $\Rightarrow$: Lemma 1.

$\Leftarrow$: $\mathcal{F}$ a coherent sheaf on X. $\mathcal{N} :=$ nilradical of X (this is an ideal sheaf)

$\exists r>0$: $\mathcal{N}^r = 0$ because the scheme is of finite type.

$\mathcal{F} \supseteq \mathcal{N}\mathcal{F} \supseteq \dots \supseteq \mathcal{N}^r\mathcal{F} = 0$. Thus we obtain short exact sequences

$0 \to \mathcal{N}^{i+1}\mathcal{F} \to \mathcal{N}^i\mathcal{F} \to \mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F} \to 0$, which in cohomology induce les

$$\dots \to H^p(X, \underbrace{\mathcal{N}^{i+1}\mathcal{F}}_{\otimes \mathcal{L}^{\otimes m}}) \to \underbrace{H^p(X, \underbrace{\mathcal{N}^i\mathcal{F}}_{\otimes \mathcal{L}^{\otimes m}}) \to H^p(X, \underbrace{\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F}}_{\otimes \mathcal{L}^{\otimes m}})}_{=0 \text{ for } p>0,\ m\gg 0} \to H^{p+1}(X, \underbrace{\mathcal{N}^{i+1}\mathcal{F}}_{\otimes \mathcal{L}^{\otimes m}}) \to \dots$$

By descending induction on i:

$$H^p(X, \mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0 \quad \forall p>0$$

2): Similar, just use the appropriate ideal sheaf instead of $\mathcal{N}$.

Pf OF THM: $\Rightarrow$: Assume D ample. $\exists m>0$, mD is very ample and defines a morphism $f: X \hookrightarrow \mathbb{P}^N$ and $\mathcal{O}_X(mD) = f^*\mathcal{O}_{\mathbb{P}^N}(1)$.

For any $Y \subseteq X$ integral subscheme: $(mD)^{\dim Y} = (mD|_Y)^{\dim Y} = \deg f(Y) > 0$. ✓

$\Leftarrow$: We may assume X to be integral by Lemmata 1 & 2.

The proof will be by induction on $\dim X =: n \geq 1$.

By induction, $D|_Y$ is ample for any subscheme $Y \subseteq X$, $\dim Y < \dim X$.

<u>Step 1.</u> $\chi(X, mD) \to \infty$ for $m \to \infty$.

$\chi(X, mD)$ is a polynomial in m of degree n with leading coeff. $\frac{D^n}{n!} > 0$.

<u>Step 2:</u> Wma D to be effective. (i.e. some multiple of D is lin. eq. to an eff. divisor.)

$\mathcal{O}_X(D) \subseteq K(X)$ is a subsheaf. Let $\mathcal{I}_1 := \mathcal{O}_X(-D) \cap \mathcal{O}_X$, $\mathcal{I}_2 := \mathcal{O}_X \cap \mathcal{O}_X(D)$,

and $Y_1, Y_2 \subseteq X$ the closed subschemes associated to $\mathcal{I}_1$ resp. $\mathcal{I}_2$.

Consider the exact sequences

$$0 \to \mathcal{I}_1(mD) \to \mathcal{O}_X(mD) \to \mathcal{O}_{Y_1}(mD) \to 0$$

$$\|$$

$$0 \to \mathcal{I}_2((m-1)D) \to \mathcal{O}_X((m-1)D) \to \mathcal{O}_{Y_2}((m-1)D) \to 0$$

$(\mathcal{O}_X(-D)\cap\mathcal{O}_X)\otimes\mathcal{O}_X(D) \simeq \mathcal{O}_X \cap \mathcal{O}_X(D)$ comes from the corresponding statement for flat modules

Aside: if X is projective, $D \sim D_1 - D_2$ where D_1 and D_2 are very ample eff. divisors.

Pf: H ample $\Rightarrow$ $D_1 := D + mH$ is very ample $\forall m \gg 0$.

Now this yields, by using long exact sequences, that

$$h^i(X, mD) = h^i(X, \mathcal{I}_1(mD)) = h^i(X, \mathcal{I}_2((m-1)D)) = h^i(X, (m-1)D) \quad \forall i \geq 2$$

because $h^j(X, \mathcal{O}_{Y_1}(mD)) = h^j(X, \mathcal{O}_{Y_2}(mD)) = 0$ by induction on $j > 0$.

It follows by Step 1 that $h^0(X, mD) - h^1(X, mD) \longrightarrow \infty$ as $m \to \infty$, replace D with mD.

Step 3. $\mathcal{O}_X(mD)$ is globally generated for $m \gg 0$.

$0 \longrightarrow \mathcal{O}_X((m-1)D) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_D(mD) \longrightarrow 0$ short exact

$H^i(D, mD) = 0 \quad \forall i > 0 \quad \forall m \gg 0$ by induction

$\Rightarrow H^i(X, \mathcal{O}_X((m-1)D)) \twoheadrightarrow H^i(X, \mathcal{O}_X(mD))$ and it must be bijective for $m \gg 0$ by looking at dimensions

$\Rightarrow H^0(X, mD) \twoheadrightarrow H^0(D, mD)$

By induction, $|mD|_D|$ is base-point-free. for $m \gg 0$, so $|mD|$ has no base point on D. But $\mathrm{Bs}|mD| \subseteq \mathrm{Supp}\, D$.

Step 4. mD is ample for $m \gg 0$

Let $f : X \longrightarrow \mathbb{P}^N$ be the morphism given by $|mD|$.

f is finite because if $\nexists\, C \subseteq X$ is a curve s.t. $f(C) = pt$ then i.e. f has a contractible fibre

$mD.C = 0$ (using $\mathcal{O}_X(mD) = f^*\mathcal{O}_{\mathbb{P}^N}(1)$) $\lightning$ by assumption, $D^{\dim Y}.Y > 0 \quad \forall Y \subseteq X$ subscheme

and use that on proper sch, qfin $\Leftrightarrow$ finite

By Lemma 1: $\mathcal{O}_X(mD)$ is ample. □

Cor. $f : X \longrightarrow Y$ finite surjective morphism of proper schemes $/k$.

If $\mathcal{L}$ is a line bundle on X s.t. $f^*\mathcal{L}$ is ample on Y then $\mathcal{L}$ is ample on X.

Pf: Could be proven w/o NM, but it's a lot easier this way.

Let $V \subseteq X$ an integral subscheme of X, W an integral subscheme of Y such that $W \overset{f}{\twoheadrightarrow} V$.

Then $0 < f^*D^{\dim W}.W = \deg(W \to V)\, D^{\dim V}.V$

f^*D is ample □

IV. Nef divisors and cones

Det. X any scheme. $\mathrm{Div}(X)_{\mathbb{Q}} := \mathrm{Div}(X) \otimes \mathbb{Q}$, $\mathrm{Div}(X)_{\mathbb{R}} := \mathrm{Div}(X) \otimes \mathbb{R}$.

These are called $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors resp. $\mathbb{R}$-Cartier divisors.

Rmk. $H \in \mathrm{Div}(X)_{\mathbb{Q}}$ is ample if an integral multiple is ample (by def)

$\Leftrightarrow H = \sum a_i A_i$, $a_i \in \mathbb{Q}_{>0}$, A_i ample Cartier divisor

$\Leftrightarrow H^{\dim V} . V > 0 \quad \forall V \subseteq X$

IV. 1. Nef divisors

Def. Let X be a proper scheme $/k$. Then $D \in \mathrm{Div}(X)_{\mathbb{Q}}$ is nef if $D.C \geq 0$ for any curve $C \subseteq X$.

numerically effective / numerically eventually free

Easy properties:

- $f: X \to Y$ morph. b/w proper schemes $/k$, D nef on $X \Rightarrow f^*D$ nef on Y (by projection formula)
- If $f: X \twoheadrightarrow Y$, f^*D nef $\Rightarrow D$ nef.

Thm. X proper scheme $/k$. If D is nef then $D^{\dim V} . V \geq 0$ for any $V \subseteq X$ integral subscheme.

Pf: Wma X to be integral. The proof is by induction on $\dim X =: n$.

The case $n=1$ is clear.

We assume that $D^{\dim V} . V \geq 0 \quad \forall V \subseteq X$ s.t. $\dim V < \dim X$.

Need to check $D^n \geq 0$.

By Chow's Lemma wma X to be projective.

Take H an ample Cartier divisor on X, and for $t \in \mathbb{R}$ let

$$p(t) := (D + tH)^n = D^n + n D^{n-1} H t + \dots + t^n H^n$$

WTS: $p(0) \geq 0$. $\nexists$ $p(0) < 0$.

$H^n > 0$ (H is ample) $\Rightarrow \exists$ maximal real root $t_0 > 0$, $p(t) > 0 \quad \forall t > t_0$

Claim. $D + tH$ is ample $\forall t > t_0$, $t \in \mathbb{Q}$

Pf: $$(D + tH)^{\dim V} . V = \underbrace{D^{\dim V} . V}_{\geq 0} + \underbrace{(\dim V) D^{(\dim V) - 1} H V t}_{\geq 0} + \dots + \underbrace{t^{\dim V} H^{\dim V} . V}_{> 0}$$

The first $\dim V$ terms are ≥ 0 since H is ample.

$\Rightarrow (D + tH)^{\dim V} . V > 0$, ample by NM.

Write $p(t) = q(t) + r(t)$ where $q(t) = D.(D+tH)^{n-1}$, $r(t) = tH.(D+tH)^{n-1}$

$q(t) \geq 0 \quad \forall t > t_0$ because $D + tH$ is ample $\Rightarrow q(t_0) \geq 0$ by taking $t \searrow t_0$

$r(t_0) = \underbrace{D^{n-1} t_0 H}_{\geq 0} + \underbrace{(n-1) D^{n-2} H^2 t_0}_{\geq 0} + \ldots + \underbrace{t_0^n H^n}_{>0}$

$\Rightarrow p(t_0) = q(t_0) + r(t_0) > 0 \quad ⨳$

Cor. X proper scheme $/k$, D nef on X, H ample. Then $D + \varepsilon H$ is ample $\forall \varepsilon > 0$, $\varepsilon \in \mathbb{Q}$.

Pf: exc., follows from one of the computations above.

IV. 2. Kleiman criterion

07.11.2018

X proj var $/k$

$Z_1(X) :=$ 1-cycles $=$ free ab gp gen'd over $\mathbb{Z}$ by curves $= \{ \sum_i e_i C_i \mid e_i \in \mathbb{Z}, C_i \text{ curve} \}$

We have an intersection pairing $\mathrm{Div}(X) \times Z_1(X) \longrightarrow \mathbb{Z}$

$D_1, D_2 \in \mathrm{Div}(X)$, $D_1 \equiv D_2$ numerical equivalent if $\forall C$ curve in X: $D_1 C = D_2 C$

Two curves C_1, C_2 are numerically equivalent if $\forall D \in \mathrm{Div}(X)$: $D_1 . C = D_2 . C$

$N^1(X) := \mathrm{Div}(X)/_\equiv \qquad N_1(X) := Z_1(X)/_\equiv \qquad N^1(X)$ is the Néron-Severi group of X

We get perfect pairings $N^1(X) \times N_1(X) \longrightarrow \mathbb{Z}$

$N^1(X)_\mathbb{R} \times N_1(X)_\mathbb{R} \longrightarrow \mathbb{R}$, same for $\mathbb{Q}$

Thm. (Severi) $N^1(X)$ is finitely generated

$\rho(X) := \dim N^1(X)_\mathbb{R}$ Picard number of X

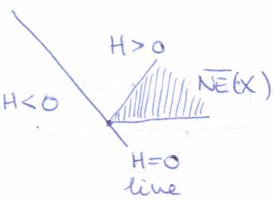
For a complex variety X: $N^1(X) \hookrightarrow H^2(X, \mathbb{Z})$

$NE(X) \subseteq N_1(X)_\mathbb{R}$ convex cone generated by curves: $NE(X) := \{ \sum_i e_i [C_i] \mid e_i \geq 0, C_i \text{ curve} \}$

$\overline{NE}(X)$ closure, called the Kleiman-Mori cone

Thm. (Kleiman's criterion) X proj var $/k$. Then a Cartier divisor D on X is ample iff $D.z > 0 \quad \forall z \in \overline{NE}(X) \setminus \{0\}$.

Cor. $\overline{NE}(X)$ does not contain any line.



Pf OF THM: Assume D to be ample.

$\nexists\ \exists z \in \overline{NE}(X)\setminus\{0\}$: $D.z = 0$.

Take a divisor $E \in N^1(X)$ s.t. $E.z < 0$. (This exists: there is an E s.t. $E.z \neq 0$ since $z \neq 0$, then change sign if necessary.)

$\Rightarrow (E + tD).z < 0 \quad \forall t > 0.$

But for $t \gg 0$, $E + tD$ is ample by ampleness of D. $\Rightarrow (E+tD)z > 0$ ↯

Now assume D to be positive on $\overline{NE}(X)\setminus\{0\}$.

Let $\|\cdot\|$ be a norm on $N_1(X)_{\mathbb{R}}$, $S := \{s \in \overline{NE}(X) \mid \|s\| = 1\}$ is compact

The function $z \longmapsto D.z$ on S is bounded from below by some constant $a > 0$

Let H be an ample divisor. The function $z \mapsto H.z$ on S is bounded from above by some constant $b > 0$.

Consider $D - \frac{a}{b}H$. This is non-negative on S by the above discussion.

$\Rightarrow$ is non-negative on $\overline{NE}(X)$ since that is just multiplying S

$\Rightarrow D - \frac{a}{b}H$ is nef $\Rightarrow \left(D - \frac{a}{b}H\right) + \frac{a}{b}H = D$ is ample by prev. exc.

IV. 3. Cones

X proj variety

Def. The ample cone $\mathrm{Amp}(X)$ is the convex cone generated by ample divisors.

Rmk. $C \subseteq V$ in an $\mathbb{R}$-vector space V is a cone if $\forall v, w \in C$: $v + w \in C$,

$\forall v \in V\ \forall \lambda \in \mathbb{R}_{>0}$: $\lambda v \in$. Note that 0 need not be in C.

Def. The nef cone $\mathrm{Nef}(X)$ is the convex cone gen'd by nef divisors.

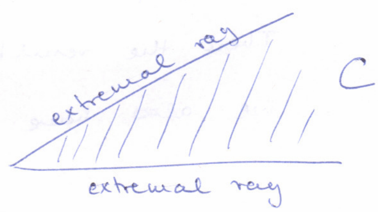
These defns make sense: sum of amples is ample etc.

Exc. $\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}$, $\mathrm{Amp}(X) = \mathrm{int}\ \mathrm{Nef}(X)$, $\overline{NE}(X)$ is the dual cone of $\mathrm{Nef}(X)$,

i.e. $\overline{NE}(X) = \{z \in N_1(X) \mid \forall \delta \in \mathrm{Nef}(X): z.\delta \geq 0\}$

Def. A subcone F of a cone $C \subseteq V$ is extremal if $\forall v, w \in C: (v + w \in F \Rightarrow v, w \in F)$.

F is also called an extremal face of C. If in addition $\dim F = 1$, it is called an extremal ray.



Here are some important facts one should already know (Hartshorne III.11.)

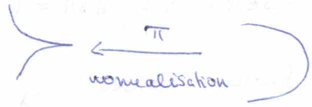
a) $f: X \to Y$ projective (works for proper too) morphism of noetherian schemes, $f_* \mathcal{O}_X = \mathcal{O}_Y \Rightarrow f^{-1}(y)$ is connected $\forall y \in Y$. (f has connected fibres)

b) Zariski's main theorem (one of the many): $f: X \to Y$ proper birational morphism of integral schemes, Y normal $\Rightarrow f_* \mathcal{O}_X = \mathcal{O}_Y$

In particular, the fibres of f are connected.

c) Stein factorisation: $f: X \to Y$ projective morphism of noetherian schemes $\Rightarrow f$ factors as $X \xrightarrow{f'} Y' \xrightarrow{\pi} Y$ where $f'_* \mathcal{O}_X = \mathcal{O}_{Y'}$ and π is a finite morphism, $Y' = \operatorname{Spec} f_* \mathcal{O}_Y$.

<u>Ex.</u> $C = \{Y^2 = X^3\} \subseteq \mathbb{A}^2$



π is birational, proj and homeo, but $\pi_* \mathcal{O}_X \neq \mathcal{O}_Y$ since X and Y are not isomorphic (one has a singular point, the other doesn't)

$\Rightarrow$ One should be careful, the above facts don't hold in non-normal cases.

<u>Ex.</u> X smooth vty $/k$, $0 < p = \operatorname{char} k$, $F: X \to Y$ Frobenius morphism is a bijective morphism but $F_* \mathcal{O}_X \neq \mathcal{O}_Y$.

$\Rightarrow$ Characteristic $p > 0$ can be more difficult than char 0.

<u>Def.</u> $\pi: X \to Y$ morph of proj varieties.

$NE(\pi) :=$ the cone in $NE(X)$ generated by contracted curves

$= \left\{ \sum_i a_i [C_i] \,\middle|\, a_i \geq 0,\ C_i \text{ integral curve on } X \text{ s.t. } \pi(C_i) = \text{point} \right\}$ relative cone of curves

<u>Rmk.</u> $C \subseteq X$ integral curve is contracted iff $\pi_* C = 0$ where $\pi_*: N_1(X)_{\mathbb{R}} \to N_1(Y)_{\mathbb{R}}$

$NE(\pi) = NE(X) \cap \operatorname{Ker}(\pi_*)$ is closed in $NE(X)$.

<u>Prop.</u> X, Y, Y' projective varieties, $\pi: X \to Y$ a morphism.

a) The subcone $NE(\pi) \subseteq NE(X)$ is extremal

b) Assume $\pi_* \mathcal{O}_X = \mathcal{O}_Y$ and let $\pi': X \to Y'$ f.w. $NE(\pi) \subseteq NE(\pi')$.
Then $\exists!\, f: Y \to Y'$: $\pi' = f \circ \pi$:

$$\begin{tikzcd} X \arrow[r, "\pi'"] \arrow[d, "\pi"'] & Y' \\ Y \arrow[ur, dashed, "\exists! f"'] & \end{tikzcd}$$

Rigidity lemma

c) π is uniquely determined by $NE(\pi)$ when $\pi_* \mathcal{O}_X = \mathcal{O}_Y$.

Pf: b)

$$\begin{tikzcd} X \arrow[r, two heads, "g = (\pi, \pi')"] \arrow[d, "\pi"'] & Z \subseteq Y \times Y' \arrow[dl, "p"] \arrow[dr, "p'"] & \\ Y & & Y' \end{tikzcd}$$

p, p' projections. Wts p is an iso.

Then the result follows: $\pi' = \underbrace{p' \circ p^{-1}}_{f} \circ \pi$, we also have uniqueness.

$y_0 \in Y$, $\pi^{-1}(y_0) = g^{-1}(p^{-1}(y_0))$ is contracted to a point by π $\Rightarrow$ also by π' $\Rightarrow$ also by $g = (\pi, \pi')$.

$g(g^{-1}(p^{-1}(y_0))) = p^{-1}(y_0)$ is hence a point $\Rightarrow$ p is injective, thus also bijective.

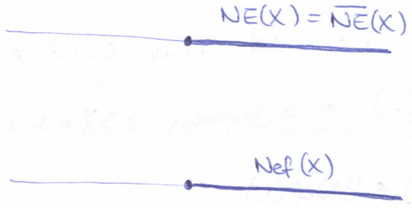
$\mathcal{O}_Y \subseteq p_* \mathcal{O}_Z \subseteq p_* g_* \mathcal{O}_X = \pi_* \mathcal{O}_X = \mathcal{O}_Y$ $\Rightarrow$ p is an iso.

(The assertions a), c) are easy.)

IV. 4. Examples.

1) X projective vty, $\rho(X) = 1$. E.g. projective spaces, curves.

$N_1(X)_{\mathbb{R}}$



$N^1(X)$

$\mathrm{id}: X \longrightarrow X$ $\qquad$ $\pi: X \longrightarrow pt$

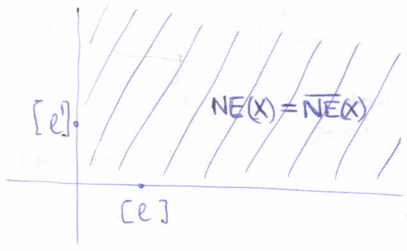
$NE(\mathrm{id}) = \{0\}$ $\qquad$ $NE(\pi) = X$

id is given by any ample divisor,

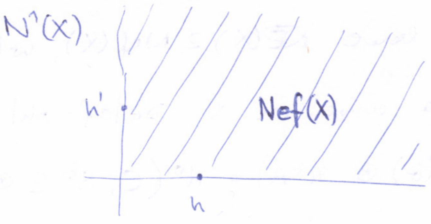
π is given by $\{0\}$

2) $X = \mathbb{P}^n \times \mathbb{P}^m$, $\rho(X) = 2$, $NE(X) = \mathbb{R}_{\geq 0}[l] + \mathbb{R}_{\geq 0}[l']$

$N_1(X)_{\mathbb{R}}$



$N^1(X)$



$$\begin{tikzcd} X \arrow[r, "\pi_2"] \arrow[dr, "\pi_1"'] & \mathbb{P}^n \\ & \mathbb{P}^m \end{tikzcd}$$

$h = \pi^* \mathcal{O}_{\mathbb{P}^n}(1)$, $\quad h' = \pi^* \mathcal{O}_{\mathbb{P}^m}(1)$

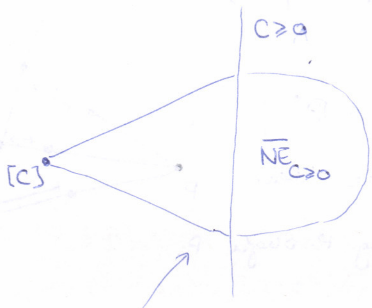
3) X smooth proj surface, $N_1(X)_{\mathbb{R}} = N^1(X)_{\mathbb{R}}$

I) $\mathrm{Amp}(X) \subseteq NE(X)$ $\Rightarrow$ $\mathrm{Nef}(X) \subseteq \overline{NE}(X)$

II) C an integral curve in X. s.t. $C^2 \leq 0$ $\Rightarrow$ $\overline{NE}(X) = \mathbb{R}_{\geq 0}[C] + \underbrace{\overline{NE}_{C \geq 0}(X)}$

the part of $\overline{NE}(X)$ that has non-negative intersection with C

III) $C^2 < 0$ $\Rightarrow$ C generates an extremal ray



section of the cone

(the 3-dimensional picture is hard to draw and to see)

An example of Mumford: let C be a smooth projective curve $/\mathbb{C}$, $g(C) \geq 2$,

E vector bundle of rk 2 on C,

$\pi : \mathbb{P}_C(E) \longrightarrow C$ is a ruled surface

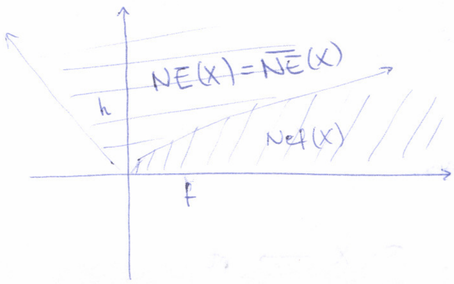
For simplicity, assume $\deg(E) = 0$

a.o.n., classes should be taken

$N^1(X)$ is generated by h and f where $h = \mathcal{O}_{\mathbb{P}(E)}(1)$, and f is the class of a fibre F.

$h^2 = \deg(E) = 0$, $h.f = 1$, $f^2 = 0$

Claim. f is on the bdry of $\overline{NE}(X)$ and $\mathrm{Nef}(X)$



clear

$(ah+f).f = a$

Case 1. E unstable, i.e. $\exists A$ line bundle, $E \twoheadrightarrow A$, $\deg A > 0$

$\Rightarrow$ $D = \mathbb{P}_C(A) \subseteq \mathbb{P}_C(E)$ is a curve, $[D] = h + af$

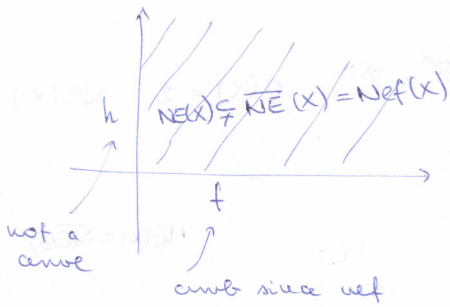
$\Rightarrow$ $C^2 = 2a < 0$ $\Rightarrow$ $[C] \in \partial\overline{NE}(X)$, $(h - af) \in \partial \mathrm{Nef}(X)$

$\overline{NE}(X) = NE(X)$

(i.e. $\nexists A$ subbundle, $\deg A > 0$)

Case 2. E semistable $\Rightarrow$ $\overline{NE}(X) = \mathrm{Nef}(X)$

but $\exists E$ s.t. mh not effective $\forall m > 0$



14.11.2018

We always have $\overline{NE}(X) \supseteq \mathrm{Nef}(X)$, wts $\subseteq$.

Let D be a curve on X, $D \in |af + bh|$, $b \geq 0$

$H^0(X, \mathcal{O}_{P/C}(b) \otimes \pi^* A) = H^0(C, S^b E \otimes A) \neq 0$

$\Rightarrow \exists\, \mathcal{O}_C \hookrightarrow S^b E \otimes A$ $\Rightarrow \exists A^{-1} \hookrightarrow S^b E$ but $S^b E$ is semistable of deg 0

$\Rightarrow \deg A \geq 0$

One can show $\exists E$: $mh \notin NE(X)$ for any $m > 0$ $\Rightarrow$ $NE \subsetneq \overline{NE}$

This also implies $h.D > 0$ $\forall D \subseteq X$ curve but h is not ample $(h^2 = 0)$

Ex. 1) $X = Bl_p \mathbb{P}^n$, $p \in \mathbb{P}^n$, $\rho(X) = 2$, $N^1(X) = \langle H, E \rangle$, $E = f^{-1}(p)$, $H = f^* \mathcal{O}_{\mathbb{P}^n}(1)$

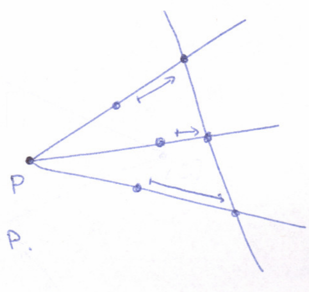
H is nef but not ample $(H.E = 0)$

because pullback of nef (in fact, pullback of ample)

Claim. $H - E$ is nef but not ample.

Pf: I) Consider the projection from P: $\mathbb{P}^n \overset{\pi}{\dashrightarrow} \mathbb{P}^{n-1}$

it's not def'd in P, and it's given by the linear system $|\mathcal{O}(1) - P|$, i.e. all the hyperplanes passing through P.



Mori theory, lecture 5

$$\begin{tikzcd} & Bl_P \mathbb{P}^n = X \arrow[dl, "f"'] \arrow[dr, "g = \text{resolution of } \pi"] & \\ \mathbb{P}^n \arrow[rr, dashed, "\pi"] & \circlearrowleft & \mathbb{P}^{n-1} \end{tikzcd}$$

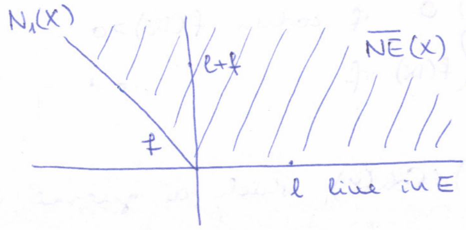
This resolution process is a general fact, but in this case it is very explicit geometrically. Points in $f^{-1}(P) = E$ parametrise lines through P in $\mathbb{P}^n$, so we know where to send each $Q \in E$.

g is given by $|H-E| \Rightarrow H-E$.

II) Let C be a curve in X.

$$(H-E).C = \begin{cases} \deg_E C > 0 & \text{if } C \subseteq E \simeq \mathbb{P}^{n-1} \\ H.C = \mathcal{O}(1).f_*(C) > 0 & \text{if } C \cap E = \emptyset \\ \mathcal{O}(1).f_*C - E.C = \deg f_*(C) - \operatorname{mult}_P f_*(C) \geq 0 & \end{cases}$$

Dual picture:



f fibre of $X \xrightarrow{g} \mathbb{P}^{n-1}$

Note. $\rho(X) = 2$ makes our life much easier b/c there are at most 2 extreme arrays. But if $\rho(X) = 3$, there can be infinitely many.

2) $X = Bl_{P_1, \dots, P_k} \mathbb{P}^2$ in k-general points, $0 \leq k \leq 8$

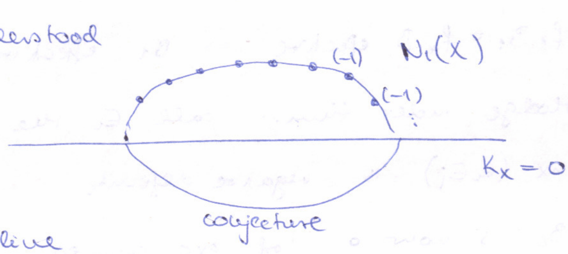
Then $-K_X$ is ample (del Pezzo), $K_X = f^* K_{\mathbb{P}^2} + E_1 + \dots + E_k$

E.g. $K_X^2 = 3 \Rightarrow X \subseteq \mathbb{P}^3$ cubic surfaces, 27 (-1)-curves (generating* extreme arrays etc)

E.g. $K_X^2 = 8 \Rightarrow 240$ (-1)-curves

*If a curve on a surface has negative self-intersection then it always generates an extreme array.

$K \geq 9 \Rightarrow X$ has ∞ (-1)-curves, not well understood



<u>Nagata's Conjecture.</u>

X normal projective vty, $V \subseteq H^0(X, \mathcal{L})$, $\mathcal{L}$ a line bundle on X, $Bs(|V|) = \bigcap_{D \in |V|} D$. We get $f: X \dashrightarrow \mathbb{P}^r$, $r = \dim V - 1$.

A priori f is not def'd on the base locus.

<u>Lemma.</u> By blowing up $Bs(|V|)$ we get a morphism

$$\begin{tikzcd} & Bl_{Bs(|V|)} X \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed] & & \mathbb{P}^r \end{tikzcd}$$

Reason: when we have a linear system, we can consider the greatest effective divisor F contained in $D_1, \dots, D_r$ (generators of V) which is the divisorial component of $Bs(|V|)$.

$|V| = |W| + |F|$ where $|W|$ has no fixed component. If $Bs(|V|) = F$ then $|W|$ is bpf

Divisorial component: irr'ble components in codim 1, with possibly non-reduced structure.

V Singularities in the MMP

From now on, $k = \mathbb{C}$.

Let $f : Y \longrightarrow X$ be a proper birat morphism of varieties.

<u>Def.</u> $\mathrm{Exc}(f) := \{x \in X \mid f \text{ is not an iso}\}$ exceptional locus of f

This is not a divisor in general.

<u>Rmk.</u> If X is normal then Zariski's Main Lemma implies that $Y \xrightarrow{f} X$ has connected fibres as $f_* \mathcal{O}_Y = \mathcal{O}_X$, so $f(\mathrm{Exc}(f))$ has codim ≥ 2.

Let D be Cartier on Y normal, X normal. Then $D = \sum c_i D_i$ is also a Weil divisor

$$f_* D := \sum c_i f_*(D_i) \quad \text{where } f_*(D_i) = \begin{cases} 0 & \text{if codim } f(D_i) > 1 \\ f(D_i) & \text{if } \quad\quad\quad\quad\quad = 1 \end{cases}$$

$f_* D$ is Weil and in general not Cartier.

Not to be mixed up with $f_*(\mathcal{O}_Y(D)) \in \mathrm{Coh}(X)$, which in general is not $\mathcal{O}_Y(f_* D)$ and not even a line bundle.

<u>Negativity Lemma.</u> $f : Y \longrightarrow X$ proper birat morphism b/w normal vties. Let $-B$ be an f-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y (i.e. $-B.C \geq 0$ for any connected curve C on Y)

Then B is effective iff $f_* B$ is effective

Pf: B eff $\Rightarrow$ $f_* B$ eff : by def of $f_*(\cdot)$.

$f_* B$ eff $\Rightarrow$ B eff : Chow Lemma $\Rightarrow$ wma f projective.

Cutting by hyperplanes wma $\dim X = \dim Y = 2$

Write $B = B_1 + B_2$ where all the components of B_1 have divisorial img and all the components of B_2 are contracted to points (we are on a surface now)

$f_* B_1 = f_* B$ effective $\Rightarrow$ B_1 effective.

Hodge index thm.: call E_i the exceptional curves. Then the intersection matrix $(E_i E_j)$ is negative definite.

B_2 is a sum of exc curves. It follows from $-B$ being f-nef that B_2 is eff. $\square$

<u>Def.</u> A normal vty X is $\mathbb{Q}$-factorial if any Weil diviser is $\mathbb{Q}$-Cartier, i.e. it has a positive multiple which is Cartier.

<u>Thm.</u> (van der Waerden's purity thm.) Let $f : Y \rightarrow X$ be as above. Assume X to be $\mathbb{Q}$-factorial (e.g. X smooth). Then any irred component of $\mathrm{Exc}(f)$ has codim 1.

Pf: Take H an ample Cartier divisor.

Then f_*H is a $\mathbb{Q}$-Cartier divisor because X is $\mathbb{Q}$-factorial.

Thus we can pull it back. Consider $f^*f_*H - H =: E$.

Note: $E \subseteq \mathrm{Exc}(f)$ because outside we have an iso so $f^*f_*H = H$ there.

$-E$ is f-ample (i.e. positive with all curves), so in part. f-nef.

Its img is $f_*(-E) = 0$, so by the negativity lemma, $+E$ is effective.

$P \in \mathrm{Exc}(f) \Rightarrow \exists$ a curve $P \in C$ contracted $\Rightarrow E.C < 0 \Rightarrow C \subseteq E \Rightarrow \mathrm{Exc}(f) \subseteq E$

Exc. X smooth, $Z \subseteq X$ smooth, $f: Y := Bl_Z X \to X$, $K_Y = f^*K_X + mE$ where $m = \mathrm{codim}_X Z - 1$.

Thm. (Ramification formula for smooth varieties)

Let $f: Y \to X$ be a proper birat morphism b/w smooth varieties.

Then $K_Y \sim f^*K_X + R$ where R is an effective divisor and $\mathrm{Supp}\, R = \mathrm{Exc}(f)$.

Pf: Local computation. 21.11.2018

Let E be a component of $\mathrm{Exc}(f)$. Purity Thm. $\Rightarrow$ codim $E = 1$.

Let $p \in E$ general and $y_1, \dots, y_n$ local coordinates for p s.t. $E = (y_1 = 0)$ locally.

$f = (f_1, \dots, f_n)$

Let $x_i := f_i(y_1, \dots, y_n)$ local coordinates for $x = f(p)$

$f^* dx_1 \wedge \dots \wedge dx_n = \mathrm{Jac}\left(\frac{f_i}{x_i}\right) dy_1 \wedge \dots \wedge dy_n = y_1^a \cdot (\text{unit}) \cdot dy_1 \wedge \dots \wedge dy_n$ where $a \in \mathbb{Z}$

$\mathrm{Jac}(p) = 0$ by the Implicit Function Thm. $\Rightarrow a > 0 \Rightarrow K_Y \sim f^*K_X + aE + \dots$

Reformulate: the pullback has a pole along the exceptional locus.

V.1.

Lemma 1. $\pi: Y \to X$ proper birat morphism, X, Y normal.

a) D Cartier on X, F an effective Cartier divisor, $F \subseteq \mathrm{Exc}(\pi)$

Then $H^0(X, D) \simeq H^0(Y, \pi^*D + F)$, i.e. $\pi_*\mathcal{O}_Y(F) = \mathcal{O}_X$.

(Note that $H^0(X, D) \simeq H^0(Y, \pi^*D)$, i.e. $\pi_*\mathcal{O}_Y = \mathcal{O}_X$ is known: the pushforward of a sheaf has the same sections as the sheaf itself.)

b) Y smooth, D, D' Cartier, F, F' ~~effective~~ exceptional divisors, $\pi^*D + F \sim \pi^*D' + F'$

$\Rightarrow D \sim D'$ and $F = F'$. (Rigidity: lin. equivalent $\Rightarrow$ equal.)

Pf: a) $H^0(X, D) \subseteq H^0(Y, \pi^* D) \subseteq H^0(Y, \pi^* D + F) \overset{?}{\subseteq} H^0(Y \setminus \mathrm{Exc}(\pi), \pi^* D + F) \simeq H^0(Y \setminus \mathrm{Exc}(\pi), \pi^* D) \simeq$

just increasing the number of sections

there was a question about why this holds, and no convincing answer was given.

$\simeq H^0(X \setminus \pi(\mathrm{Exc}(\pi)), D) = H^0(X, D)$

$\pi(\mathrm{Exc}(\pi))$ has codim ≥ 2, hence $g|_{X \setminus \pi \mathrm{Exc}\pi} + D|_{X \setminus \pi \mathrm{Exc}\pi} \geq 0 \Rightarrow g + D \geq 0$

b) $F = F_1 - F_2$, $F' = F_1' - F_2'$ where F_1, F_2, F_1', F_2' are eff exceptional with no common components.

$$\pi^*(D - D') + F_1 + F_2' \sim F_2 + F_1'$$

$$\overset{a)}{\Longrightarrow} \quad D - D' \sim 0$$

$$\Longrightarrow \quad F_1 + F_2' \sim F_2 + F_1'$$

$\square$

If X is a smooth vty then Ω^1_X is free and $\omega_X = \Lambda^n \Omega^1_X$ is an invertible sheaf, K_X is a divisor s.t. $\omega_X \simeq \mathcal{O}_X(K_X)$

If X is normal then $\mathrm{codim}_X \mathrm{Sing}(X) \geq 2$, $\mathrm{Cl}(X) = \mathrm{Cl}(X_{reg})$ where $X_{reg} = X \setminus \mathrm{Sing}(X)$

$K_{X_{reg}}$ extends to a Weil divisor K_X on X

$\mathcal{O}_X(K_X)|_U = \{ f \in K^\times \mid \mathrm{div}\, f|_U + K_X|_U \geq 0 \}$ is invertible iff K_X is Cartier.

If K_X is $\mathbb{Q}$-Cartier and $f : Y \longrightarrow X$ is a resolution of singularities (i.e. Y smooth, f proper birational) then $K_Y \underset{\mathbb{Q}}{\sim} f^* K_X + E$ where E is an exceptional divisor.

Def. Let X be a normal vty s.t. K_X is $\mathbb{Q}$-Cartier.

1) X is said to have terminal singularities if for any resolution $f : Y \longrightarrow X$ one can write $K_Y \underset{\mathbb{Q}}{\sim} f^* K_X + E$ where E is effective and supp E contains all the exceptional divisors. (Consequently, E itself is exceptional.)

2) X is said to have canonical singularities if for any resolution $f : Y \longrightarrow X$ one has $K_Y \underset{\mathbb{Q}}{\sim} f^* K_X + E$ where E is exceptional and effective.

Note that 2) $\Leftarrow$ 1).

Rmk. X smooth $\Rightarrow$ X terminal (i.e. has terminal singularities) $\Rightarrow$ X canonical.

Note that while K_X is def'd only up to lin equivalence, E is well-def'd: changing K_X does not change E, as seen in Lemma 1.

Rmk. It's enough to check only one resolution in the def. because of the ramification formula for smooth varieties: if $Y \longrightarrow X$, $Y' \longrightarrow X$ are resolutions, take W s.t.

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] & \\ Y \arrow[dr] & & Y' \arrow[dl] \\ & X & \end{tikzcd}$$

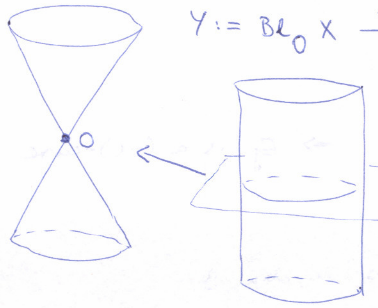
and it suffices to check at W.

Rmk. Terminal $\overset{L1.}{\iff}$ if $f: Y \longrightarrow X$ resolution and $m \in \mathbb{Z}$ s.t. mK_X is Cartier then
$f_* \mathcal{O}_Y(mK_Y - E) \simeq \mathcal{O}_X(mK_X)$ for any exceptional reduced divisor.

Canonical $\overset{L1.}{\iff}$ if $f: Y \longrightarrow X$ resolution and $m \in \mathbb{Z}$, mK_X Cartier then
$f_* \mathcal{O}_Y(mK_Y) \simeq \mathcal{O}_X(mK_Y)$.

Warning. Assume $K_Y \sim f^* K_X + E$, E exceptional. Then $f_* K_Y \sim K_X$ as divisors is always true. But $f_* \mathcal{O}_Y(K_Y) \not\simeq \mathcal{O}_X(K_X)$ if E is not effective. because $f_* \mathcal{O}_Y(E) \subsetneq \mathcal{O}_X$

Examples. 1) $\{X = x^2+y^2+z^2 = 0\} \subseteq \mathbb{A}^3$ cone over rational conic $C = \{x^2+y^2+z^2=0\} \subseteq \mathbb{P}^2$



$Y := Bl_0 X \xrightarrow{f} X$, kind of like a cylinder, Y smooth

$Exc(f) = E$ curve isomorphic to C

$K_Y = f^* K_X + aE$, a is called the discrepancy

$\downarrow$ intersect with E

$E.K_Y = (f^* K_X + aE).E = aE^2 = \underset{\text{degree of the curve}}{-2a}$

$(K_Y + E).E = 2g(E) - 2 = -2$
$(K_Y + E).E = K_Y.E - 2$
$\Big\} \Rightarrow K_Y.E = 0 \Rightarrow a = 0$

(use adjunction formula)

X has canonical but not terminal singularities

2) $X = \{x^3+y^3+z^3 = 0\} \subseteq \mathbb{A}^3$ cone over an elliptic curve. Similar computations as in 1).
$K_Y \sim f^* K_X - E$ not canonical. (log-canonical)

3) $V \subseteq \mathbb{P}^5$ the Veronese surface, $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$, $[x,y] \mapsto [x^2, y^2, z^2, xy, xz, yz]$

$X :=$ the cone over V in $\mathbb{A}^6$

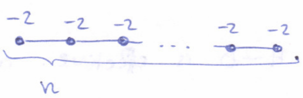
$Y := Bl_0 X$, $K_Y \sim f^* K_X - aE$, $E \simeq V$ exceptional, $E|_E \simeq \mathcal{O}_{\mathbb{P}^2}(-2)$,

$(K_Y + E)|_E = K_E \simeq \mathcal{O}_{\mathbb{P}^2}(-3) \Rightarrow a = \frac{1}{2}$, X has terminal singularities

Later: terminal singularities are somehow the minimal singularities one needs to consider to run an MMP.

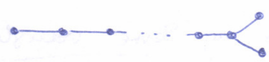
Thm. In dimension 2, canonical singularities (also called du Val singularities or rational double points) are classified locally as follows:

A_n: $z^2 + x^2 + y^{n+1} = 0$, $n \geq 1$



Dynkin diagram

D_n: $z^2 + y(x^2 + y^{n-2}) = 0$, $n \geq 4$



E_6
E_7 $\Big\}$ exceptional ones.
E_8

Prop. In dimension 2, terminal singularities are smooth.

Pf: Let X be a surface with terminal singularities and $f: Y \longrightarrow X$ resolution.

Then $K_Y \sim f^* K_X + \sum a_i E_i$, $\forall a_i > 0$, $\forall E_i$ is exceptional.

Wma that there are no (-1)-curves contained in $\mathrm{Exc}(f)$: otherwise contract it and still have a resolution.

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & Z \arrow[d] \\ & X \end{tikzcd}$$

Y is called minimal resolution

X not smooth $\Rightarrow$ there are E_i's.

Hodge index thm. $\Rightarrow (E_i . E_j)$ is neg. definite.

$K_Y . \sum a_i E_i = \left(f^* K_X + \sum a_i E_i\right).\left(\sum a_i E_i\right) = \left(\sum a_i E_i\right)^2 < 0$ since $\forall a_i < 0$. and HIT

$\Rightarrow \exists j: K_Y . E_j < 0$

$$\left.\begin{array}{l} \underbrace{(K_Y + E_j). E_j}_{<0 \text{ by HIT}} = \underset{\nwarrow \text{see Ex. 1)}}{2g(E_j) - 2} \end{array}\right\} \Rightarrow g(E_j) = 0 \Rightarrow K_Y . E_j = -1 = E_j^2 \Rightarrow E_j \text{ is a } (-1)\text{-curve } ↯$$

□

Rmk. Mori–Reid gave a complete classification of terminal singularities in dim 3.
(In higher dimension, this is hopeless to do.)

$cA_n: \; x^2 + y^2 + z^2 + w^{n+1} = 0, \quad n \geq 1$

$cD_n: \; \ldots$ then quotient by finite groups,

and this gives all the terminal singularities in dim 3.

V.2 Big divisors

28.11.2018

Def. X normal complete variety (recall that we are always over $\mathbb{C}$), D Cartier on X.
Then the Iitaka dimension of D is $k(D) := \begin{cases} \infty & \text{if } |mD| = \emptyset \quad \forall m \geq 1 \\ \max\limits_{m \geq 1} \{\dim \varphi_{|mD|}(X)\} & \text{otw.} \end{cases}$

If $k(D) = \dim X$ then D is called big.

Def. The semigroup of D is $S(D) := \{ m \in \mathbb{N} \mid h^0(mD) \neq 0\}$
The exponent of D is $e(D) := \gcd(S(D))$

Ex. Enrique's surface S, K_S is torsion, $S(D) = 2\mathbb{N}$

Lemma. X as above, D Cartier on X. Then $\exists a, b > 0$: $a m^{k(D)} \leq h^0(X, mD) \leq b m^{k(D)}$
$\forall m \in S(D)$, $m \gg 0$. (or equivalently, m large and divisible enough)

Pf: When D is big.

Let H be ample on X. s.t. $H - D$ is effective. and $h^i(X, mH) = 0 \; \forall m \geq 1 \; \forall i > 0$

$H := m_0 H_0 + D$ for some H_0 ample, $m_0 \gg 0$

(we can ask for the latter by Serre vanishing)

$\Rightarrow h^0(X, mD) \leq h^0(X, mH) = P(X, H) = \frac{H^n}{n!} m^n + O(m^{n-1})$ Hilbert polynomial, $H^n > 0$

works for any Cartier divisor D

$\Rightarrow h^0(X, mD) \leq b m^n$ for suitable b.

by def: look at the global sections

here we use $m \in S(D)$

Other direction: $\varphi_{|D|}: X \to Y \subseteq \mathbb{P}^N$ up to multiple of D, $\dim Y = \dim X$. (by bigness)

$h^0(Y, \mathcal{O}_Y(m)) = \underbrace{\frac{\deg Y}{n!}}_{=:a} m^n + O(m^{n-1})$, $\varphi^*: H^0(Y, \mathcal{O}_Y(m)) \hookrightarrow H^0(X, mD)$

≥ 0: by RR, the coeffs are intersections, which are positive by NM.

<u>Kodaira's Lemma.</u> D big Cartier, F effective divisor on X.

$\Rightarrow h^0(X, mD - F) \neq 0 \quad \forall m \in S(D), \ m \gg 0.$

Pf: Exact sequence + tensoring

$$0 \to \mathcal{O}_X(mD-F) \to \mathcal{O}_X(mD) \to \mathcal{O}_F(mD) \to 0$$

$$0 \to H^0(X, mD-F) \to \underbrace{H^0(X, mD)}_{m^n \lesssim} \to \underbrace{H^0(F, mD|_F)}_{m^{n-1} \lesssim} \to H^1(X, mD-F)$$

$\Rightarrow H^0(X, mD-F) \neq 0$ for $m \gg 0$, $m \in S(D)$.

<u>Prop.</u> X proper normal vty, D Cartier. TFAE:

(1) D is big

(2) for any ample divisor A: $mD \sim A + E$ for some $m > 0$, $E \geq 0$.

(3) $\varphi_{|mD|}$ is birational for $m \in S(D)$, $m \gg 0$.

Pf: The only implication which is not clear is (1) $\Rightarrow$ (2).

Let A be ample, $m \gg 0$ s.t. $mA \geq 0$, $(m+1)A \geq 0$.

Kodaira's Lemma $\Rightarrow \exists F \geq 0$: $mD \sim (m+1)A + F \sim A + \underbrace{(mA + F)}_{\geq 0}$

<u>Cor.</u> D big $\Leftrightarrow mD \equiv A + E$, A ample, $E \geq 0$.

One direction is clear. For the other, use that amplitude is a numerical property (by Kleiman).

V. 3 Canonical models

<u>Def.</u> X normal proper vty s.t. K_X is $\mathbb{Q}$-Cartier. Then $k(X, K_X) =: k(X)$ is the Kodaira dimension

<u>Exc.</u> D big $\Rightarrow e(D) = 1$.

<u>Def.</u> X is of general type if K_X is big, i.e. $k(X) = \dim X$.

<u>Rmk.</u> X, Y proper birational varieties with canonical singularities $\Rightarrow h^0(X, mK_X) = h^0(Y, mK_Y)$ $\forall m \geq 0$. In particular, $R(X, K_X) \simeq R(Y, K_Y)$ where $R(X, K_X) := \bigoplus_{m \geq 0} H^0(X, mK_X)$ is the canonical ring of X.

<u>Def.</u> X smooth proper*. Then $X_{can} := \operatorname{Proj} R(X, K_X)$ is the canonical model of X.

* We need the additional assumption that $R(X, K_X)$ is fin gen in order for X_{can} to be a variety. Or use the following

<u>Thm.</u> (BCHM 2010) X smooth projective vty $\Rightarrow R(X, K_X)$ is fin. gen.

Prop. X proper normal vty, D Cartier. Assume that $\bigoplus_{m\geq 0} H^0(X, mD)$ is generated by $H^0(X,D)$.
Let $Z := \mathrm{Im}\, \varphi_{|D|}(X) \subseteq \mathbb{P}^N$, and H a hyperplane section of Z. Assume D to be big.
Then 1) φ is birational
2) Z is normal
3) every divisor in $\mathrm{Bs}\,|D|$ is contracted by φ
4) $\varphi_* D \sim H$
5) if D is nef then $|D|$ is bpf.

Pf: Resolve φ:

$$\begin{tikzcd} & X' \arrow[dl, "\pi"'] \arrow[dr, "\varphi'"] & \\ X \arrow[rr, dashed, "\varphi"'] & & Z \end{tikzcd}$$

where π is a birational morphism.
(See p. 17, Lemma for this construction.)

$D' := \pi^* D$, $|D'| = |M| + F$ where $M := \varphi'^* H$ is bpf, $F \geq 0$, $F = \mathrm{Bs}\,|D'| = \pi^{-1}\,\mathrm{Bs}\,|D|$

Since $R(X,D)$ is generated in degree 1 by $H^0(X,D)$, we have

$$H^0(X', mD') = \underbrace{H^0(X, mD)}_{\sim m^n} = H^0(Z, mH)$$

$\Rightarrow$ $\dim Z = \dim X$, φ' and φ are generically finite

Stein factorisation: $X' \longrightarrow Z' \xrightarrow{p} Z$ where Z' is normal and p is finite.

$$H^0(Z, mH) \subsetneq H^0(Z', m p^* H) \subseteq H^0(X', mD') = H^0(X, mH)$$

H is ample $\Rightarrow$ $Z' = Z$. In particular, $\underbrace{Z \text{ is normal}}_{2)}$ and $\underbrace{\varphi \text{ is birational}}_{1)}$.

see Prop. (3) on p. 23

3): $\lightning$ $\exists\, G$ component of F s.t. $\varphi'(G)$ is a divisor.

$0 \to \mathcal{O}_X(\ell M) \longrightarrow \mathcal{O}_X(\ell M + G) \longrightarrow \mathcal{O}_G(\ell M + G) \to 0$ induces a les in cohomology:

$$0 \to H^0(X, \ell M) \to H^0(X, \ell M + G) \xrightarrow{\sigma} H^0(G, \ell M + G) \to H^1(X, \ell M)$$

$G \subseteq F = \mathrm{Bs}\,|D'|$ $\Rightarrow$ $\sigma = 0$ $\Rightarrow$ $h^1(X, \ell M) \geq h^0(G, \ell M + G) \sim \ell^{n-1}$ because $\varphi'(G)$ is a divisor and hence $M|_G$ is big.

By Leray's spectral sequence:

$$0 \to \boxed{H^1(Z, \varphi'_* \mathcal{O}_X(\ell M))} \to H^1(X, \mathcal{O}_X(\ell M)) \xrightarrow{\simeq} H^0(Z, R^1\varphi'_* \mathcal{O}_X(\ell M)) \to$$
$$\to \boxed{H^2(Z, \varphi'_* \mathcal{O}_X(\ell M))} \to$$

These are 0 for $\ell \gg 0$ by Serre vanishing

R^1 is supported on the locus where the fibre has dim ≥ 1, which has codim ≥ 2 since Z is normal.

$\Rightarrow$ $H^0(Z, R^1\varphi'_* \mathcal{O}_X(\ell M)) \lesssim \ell^{n-2}$ $\lightning$

4): $\varphi'_* D' = \varphi_* D \sim H$

5): D nef $\Rightarrow$ D' nef $\Rightarrow$ $F = 0$ by the negativity lemma.

Cor. D big and nef and $\underbrace{R(X,D) \text{ is fin gen}}$ $\Rightarrow$ $|D|$ is bpf.

Note that this doesn't always hold.

Thm. (Reid 1980) X smooth proper vty of general type s.t. $R(X, K_X)$ is fin gen.

Then 1) $X_{can} = \text{Proj } R(X, K_X)$ is a normal proj vty birational to X.

2) $K_{X_{can}}$ is $\mathbb{Q}$-Cartier and ample

3) X_{can} has canonical singularities.

4) if K_X is nef then $X \dashrightarrow X_{can}$ is a morphism.

Pf: Take $D = rK_X$ where $r>0$ s.t. $H^0(X, rK_X)$ generates $R(X, K_X)$

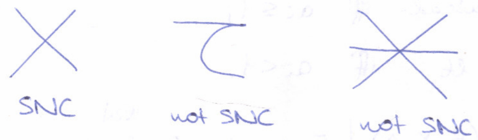
Apply Prop. 3): $rK_Z \sim \varphi_*(rK_X) \sim \varphi_*(D) \sim H$

Since $H^0(X, \mathcal{O}_X(mK_X)) \simeq H^0(X_{can}, \mathcal{O}_{X_{can}}(mK_{can}))$ for $m \geq 0$, X_{can} has canonical singularities.

V.4 Singularities of pairs

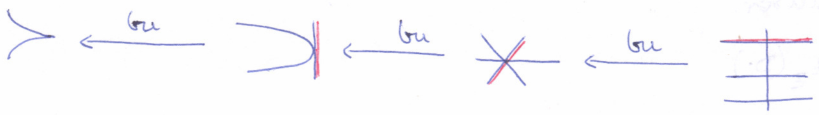
Def. X smooth vty, $D = \sum_i a_i D_i$ $\mathbb{Q}$-divisor (Weil). Then D is SNC if each D_i is smooth and they intersect transversally, i.e. $\forall p \in X$ $\exists$ local coordinates $x_1, \dots, x_n$ s.t.

$p \in D_{i_1} \cap \dots \cap D_{i_n} = \{x_{i_1} = \dots = x_{i_n} = 0\}$



Def. X normal vty, D Weil $\mathbb{Q}$-divisor. A log resolution of (X, D) is a proper birational morphism $f: Y \to X$ s.t. Y is smooth and $\text{Exc}(f) \cup \text{Supp}(f^{-1}D)$ is SNC.

Ex. $X = \mathbb{A}^2$, $D = \succ$



Def. A log pair (X, Δ) consists of a normal vty X and Weil $\mathbb{Q}$-divisor Δ.

Usually $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

Def. Δ is called a boundary if $\Delta = \sum a_i D_i$, $0 \leq a_i \leq 1$

Iitaka used log pairs to study non-complete varieties $U = X \setminus \Delta$, $X = \overline{U}$.

$K_X + \Delta$ behaves similarly to K_X.

Logarithmic forms: X smooth, $\Delta = \sum D_i$ SNC. Then a local generator of $\mathcal{O}_X(K_X + \Delta)$ looks like $\sigma = dx_1 \wedge \dots \wedge dx_n$.

$\Delta = \sum a_i D_i, \quad a_i \in \mathbb{Q}$

Def. $f: Y \to X$ proper birational, Y smooth. $f_*^{-1}\Delta := \sum a_i f_*^{-1} D_i$ is the strict transform.

$$\mathcal{O}_Y\left(m\left(K_Y + f_*^{-1}\Delta\right)\right)\Big|_{Y \setminus Exc(f)} \cong f^* \mathcal{O}_X\left(m\left(K_X + \Delta\right)\right)\Big|_{Y \setminus Exc(f)}$$

$$K_Y + f_*^{-1}\Delta \underset{\mathbb{Q}}{\sim} f^*(K_X + \Delta) + \sum_{E \text{ exc.}} a(E, X, \Delta) \cdot E \quad \Rightarrow K_Y \underset{\mathbb{Q}}{\sim} f^*(K_Y + \Delta) + \sum_{\substack{E \subseteq Y \\ \text{div}}} a(E, X, \Delta) \cdot E$$

Def. $a(E, X, \Delta)$ is the discrepancy of E wrt. (X, Δ)

$a(D_i, X, \Delta) = -a_i, \quad a(D, X, \Delta) = 0 \quad \forall D \subseteq X, \ D_i \neq D$

Rmk. $a(E, X, D)$ doesn't depend on f

$$\begin{tikzcd} Y \arrow[rr, dashed, "\varphi"] \arrow[dr, "f"'] & & Y' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

s.t. $E' = \varphi(E) \subseteq Y'$ is a divisor

φ induces an iso at the generic point of E

More intrinsically: $a(E, X, D)$ depends only on the valuation $\mathrm{ord}_E : K^\times(X) \setminus \mathbb{Z}$

Def. (X, Δ) log pair, $\Delta \geq 0$. Then (X, Δ) is klt (Kawamata log terminal) if $\forall E$ divisor over X: $a(E, X, \Delta) > -1$.

(X, Δ) is lc (log canonical) if $\forall E$ divisor over X: $a(E, X, \Delta) \geq -1$

Ex. $\dim X = 1$, $\Delta = \sum a_i P_i$. Then (X, Δ) is log canonical iff $a_i \leq 1$;

Kawamata lt iff $a_i < 1$.

Def. (X, Δ) log pair. Discrepancy: $\mathrm{discr}(X, \Delta) = \inf\{a(E, X, \Delta) \mid E \text{ exceptional Weil divisor over } X\}$

Total discrepancy: $\mathrm{tot.discr}(X, \Delta) = \inf\{a(E, X, \Delta) \mid E \text{ any divisor over } X\}$

The lower the discrepancy, the more singular (X, Δ) is.

Lemma 1. X smooth, $\Delta = \sum_i a_i D_i$, $Z \subseteq X$ closed smooth subvariety of codim k.

$\pi: Bl_Z X \longrightarrow X$, E exceptional divisor.

Then $a(E, X, \Delta) = k - 1 - \sum_i a_i \cdot \mathrm{mult}_Z(D_i)$

Lemma 2. $f: Y \to X$ birational proper, Δ_Y, Δ_X $\mathbb{Q}$-divisors s.t. $(X, \Delta_X), (Y, \Delta_Y)$ are log pairs.

If $K_Y + \Delta_Y = f^*(K_X + \Delta_X)$ then $a(E, X, \Delta_X) = a(E, Y, \Delta_Y)$.

Prop. Assume X to be smooth and $\Delta = \sum a_i D_i$ s.t. $\sum D_i$ is SNC.

Then $\mathrm{discr}(X, \Delta) = \min\left(\min\{1 - a_i - a_j \mid i \neq j, D_i \cap D_j \neq \emptyset\}, \min\{1 - a_i \mid i\}, 1\right)$.

Pf: $r(X, \Delta) := \mathrm{RHS}$.

- $\mathrm{discr}(X, \Delta) \leq r(X, \Delta)$:

  Blowup $Z \subseteq X$ s.t. $Z \not\subseteq D_i \ \forall i$ and codim $Z = 2 \Rightarrow a(E, X, \Delta) = 1$

  Blowup $Z \subseteq X$ s.t. $Z \subseteq D_i$ for some i and codim $Z = 2 \Rightarrow a(E, X, \Delta) = 1 - a_i$

  Blowup $Z \subseteq X$ s.t. $Z \subseteq D_i \cap D_j$ for some i, j and codim $Z = 2 \Rightarrow a(E, X, \Delta) = 1 - a_i - a_j$.

$\Rightarrow$ $\operatorname{discr}(X,\Delta) \le r(X,\Delta)$.

Note that $\nexists Z$: $Z \subseteq D_i \cap D_j \cap D_k$ of codim 2 because any such Z has codim $\geqq 3$.

- $\operatorname{discr}(X,\Delta) \geqslant r(X,\Delta)$: let E be an exc divisor for $f: Y \to X$. Nts $a(E,X,\Delta) \geqq r(X,\Delta)$

Up to shrinking X and Y, wma $f(E)$ is smooth in X.

$X_1 \xrightarrow{f_1} X$ blowup along $f(E)$

UP of blowup $\Rightarrow$
$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[dr, "f"'] & X_1 \arrow[d, "f_1"] \\ & X \end{tikzcd}$$
g is the composition of $(t-1)$ blowups along smooth loci

$f(E) \subseteq D_i \Leftrightarrow i \le b$ for some $b \leqq k = \operatorname{codim} f(E)$

$E_1 \subseteq X_1$ exc div

$a(E_1, X, \Delta) \geqslant r(X,\Delta)$ by the same computation as before

If $t=1$, we are done.

If $t \geqslant 2$: let Δ_1 be s.t. $K_{X_1} + \Delta_1 = f_1^*(K_X + \Delta)$

$r(X_1, \Delta_1) \geqslant r(X,\Delta)$ (computation omitted)

Induction on t: $a(E, X, \Delta) \geqslant r(X_1,\Delta_1) \geqslant r(X,\Delta)$.

Cor. (X,Δ) log pair, $\Delta \geqslant 0$, $f: Y \to X$ log resolution of (X,Δ), $K_Y \sim_{\mathbb{Q}} f^*(K_X+\Delta) + \sum_i a_i(E) E$

Y smooth, $f_*^{-1}\Delta \cup \operatorname{Exc}(f)$ is SNC

Then (X,Δ) is Klt $\Leftrightarrow$ $a_i(E) > -1$

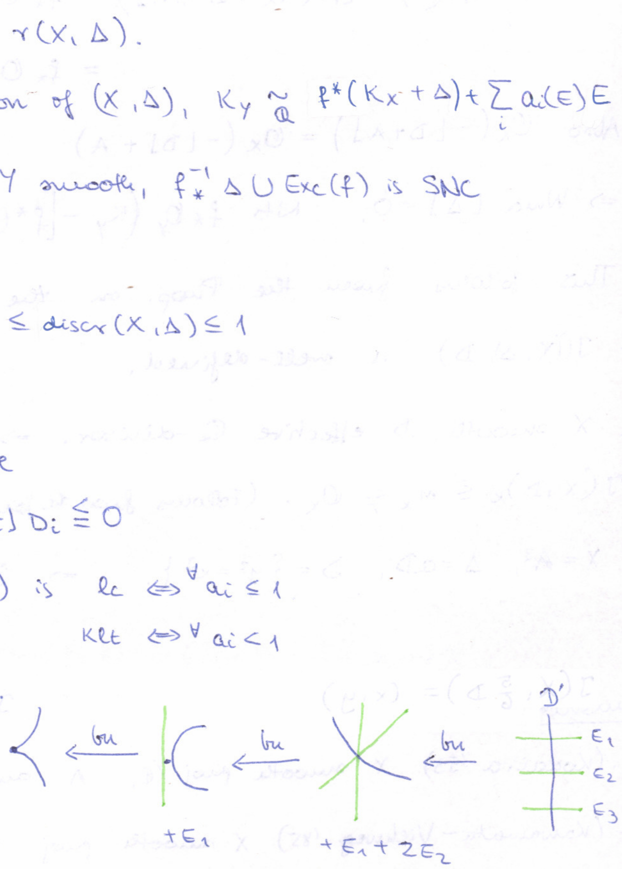
(X,Δ) is lc $\Leftrightarrow$ $a_i(E) \geqq -1$

Exc. Either $\operatorname{discr}(X,\Delta) = -\infty$ or $-1 \le \operatorname{tot.discr}(X,\Delta) \le \operatorname{discr}(X,\Delta) \le 1$

Rmk.
- (X,Δ) lc $\Rightarrow$ $0 \le a_i \le 1$
- (X,Δ) lc $\Leftrightarrow$ $\operatorname{discr}(X,\Delta) \geqslant -1$ for Δ effective
- (X,Δ) Klt $\Leftrightarrow$ $\operatorname{discr}(X,\Delta) > -1$ and $\lfloor \Delta \rfloor := \sum \lfloor a_i \rfloor D_i \leqq 0$
- X smooth, $\sum D_i$ SNC, $a_i \geqslant 0$. Then $(X, \sum a_i D_i)$ is lc $\Leftrightarrow \forall a_i \le 1$

  Klt $\Leftrightarrow \forall a_i < 1$
- Cone over elliptic curve is lc but not Klt.

Ex. $X = \mathbb{A}^2$, $\Delta = aD$, $D = \{x^2 = y^3\} \subseteq \mathbb{A}^2$ (not SNC)



$0 \le a \le 1$, $K_Y = f^*K_X + E_1 + 2E_2 + 4E_3$

$$f^*D = \underbrace{f_*^{-1}D}_{D'} + 2E_1 + 3E_2 + 6E_3$$

$$K_Y = f^*(K_X + aD) - aD' + (1-2a)E_1 + (2-3a)E_2 + (4-6a)E_3$$

$\Rightarrow$ If $\frac{5}{6} < a \le 1$ then (X, aD) not lc,

$a = \frac{5}{6}$ then (X,aD) lc but not Klt,

$0 \le a < \frac{5}{6}$ then (X,aD) Klt.

VI Multiplier ideal sheaves

Def. (X,Δ) log pair, D $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, $f: Y \to X$ log resolution of $(X, \Delta + D)$.

Write $K_Y \sim_{\mathbb{Q}} f^*(K_X + \Delta + D) + \sum a_i E_i$.

The multiplier ideal associated to X, Δ, D is

$$\mathcal{J}((X,\Delta), D) := f_* \mathcal{O}_Y(\lceil \sum a_i E_i \rceil) = f_* \mathcal{O}_Y(K_Y - \lfloor f^*(K_X + \Delta + D) \rfloor)$$

Rmk. $\mathcal{J}((X,\Delta), D) =: \mathcal{J}(X, \Delta + D)$

If $\Delta, D \geq 0 \Rightarrow \mathcal{J}(X,\Delta) \subseteq \mathcal{O}_X$

(X,Δ) klt $\Leftrightarrow \mathcal{J}(X,\Delta) = \mathcal{O}_X$

(X,Δ) lc $\Leftrightarrow \forall \varepsilon \in (0,1]: \mathcal{J}(X, (1-\varepsilon)\Delta) = \mathcal{O}_X$

For $\Delta \geq 0$: $\text{Non-klt}(X,\Delta) := \text{Supp}\left(\mathcal{O}_X / \mathcal{J}(X,\Delta)\right)$ non-klt locus

Lemma. X smooth, D $\mathbb{Q}$-divisor with SNC support, $f: Y \to X$ a log resolution of (X,D).

Then $f_* \mathcal{O}_Y(K_Y - \lfloor f^*(K_X + D) \rfloor) = \mathcal{O}_X(-\lfloor D \rfloor)$

Pf: Let A be an integral Cartier divisor on X.

Then $f_* \mathcal{O}_Y(K_Y - \lfloor f^*(K_X + D + A) \rfloor) = f_* \mathcal{O}_Y(K_Y - f^*A - \lfloor f^*(K_X + D) \rfloor)$

$= f_* \mathcal{O}_Y(K_Y - \lfloor f^*(K_X + D) \rfloor) \otimes \mathcal{O}_X(-A)$

Also $\mathcal{O}_X(-\lfloor D + A \rfloor) = \mathcal{O}_X(-\lfloor D \rfloor - A)$

$\Rightarrow$ Wma $\lfloor D \rfloor = 0$. Nts $f_* \mathcal{O}_Y(K_Y - \lfloor f^*(K_X + D) \rfloor) = \mathcal{O}_X$

This follows from the Prop. on the discrepancies b/c $\text{discr}(X,D) = \min(\dots) > -1$. □

Cor. $\mathcal{J}((X,\Delta), D)$ is well-defined. □

Exc. X smooth, D effective $\mathbb{Q}$-divisor. Then if $\text{mult}_x D \geq \dim X$ for some $x \in X$ then

$\mathcal{J}(X,D)_x \subseteq \mathfrak{m}_x \subsetneq \mathcal{O}_x$ (Follows from the formula with blowup)

Ex. $X = \mathbb{A}^2$, $\Delta = aD$, $D = \{y^2 = x^3\}$. $\Rightarrow \mathcal{J}(X, aD) \begin{cases} = \mathcal{O}_X & \text{if } a < \frac{5}{6} \\ \subsetneq \mathcal{O}_X & \text{if } a \geq \frac{5}{6} \end{cases}$

$\mathcal{J}(X, \frac{5}{6} D) = (x,y)$

Vanishing theorems

19.12.2018

Thm. (Kodaira '53) X smooth proj $/\mathbb{C}$, A ample lb. $\Rightarrow H^i(X, K_X + A) = 0 \ \forall i > 0$

Thm. (Kawamata–Viehweg '85) X smooth proj, L nef big $\mathbb{Q}$-divisor, $\lceil L \rceil - L$ has SNC support.

$\Rightarrow H^i(X, K_X + \lceil L \rceil) = 0 \ \forall i > 0$.

Special case: L is Cartier (integral) $\Rightarrow H^i(X, K_X + L) = 0 \ \forall i > 0$.

(As $\lceil L \rceil = L$, the SNC condition always holds.)

Lemma. (Local vanishing) Let $f: X \to Y$ be a log resolution of a pair (X, Δ).
Then $R^i f_* \underbrace{\mathcal{O}_Y(K_Y - \lfloor f^*(K_X+\Delta) \rfloor)}_{\mathcal{J}(X,0)} = 0 \quad \forall i>0$

Pf: WMA X, Y to be projective. This is not difficult but not trivial either. (Omitted.)

Take A an ample div on X s.t. $mA - (K_X+\Delta)$ is ample $\forall m>0$,

$R^i \mathcal{J}(X,0) \otimes \mathcal{O}_X(mA)$ is glob gen $\forall m>0 \ \forall i>0$,

$H^i(X, R^i\mathcal{J}(X,0) \otimes \mathcal{O}_X(mA)) = 0 \quad \forall m>0 \ \forall i>0$.

Leray $\Rightarrow$ $H^0(X, R^i\mathcal{J}(X,0) \otimes \mathcal{O}_X(mA)) = H^i(Y, \mathcal{O}_Y(K_Y - \lfloor f^*(K_Y+\Delta)\rfloor) \otimes \mathcal{O}_Y(f^*(mA)))$

$= H^i(Y, K_Y + \lceil f^*(mA - K_X - \Delta)\rceil)$

$= 0$ by Kawamata–Viehweg vanishing

Since all sections $H^0(\dots) = 0$, $R^i\mathcal{J}(X,0)$ must be trivial too.

Thm. (Nadel vanishing) (X,Δ) log pair, L Cartier div on X s.t. $L-(K_X+\Delta)$ is nef and big.
$\Rightarrow H^i(X, \mathcal{J}(X,\Delta) \otimes \mathcal{O}_X(L)) = 0 \quad \forall i>0$.

Cor. If (X,Δ) is klt then $\mathcal{J}(X,\Delta) = \mathcal{O}_X \Rightarrow H^i(X,L) = 0$ whenever $L = K_X + \Delta + (\text{nef big})$.

PF OF THM: Let $f: Y \to X$ be a resolution.

Local vanishing $\Rightarrow R^i(f_*\mathcal{O}_Y(K_Y - \lfloor f^*(K_X+\Delta)\rfloor + f^*L) = 0 \quad \forall i>0$

Leray $\Rightarrow H^i(Y, K_Y - \lfloor f^*(K_Y+\Delta)\rfloor + f_*L) = H^i(X, \mathcal{J}(X,\Delta)\otimes \mathcal{O}_X(L))$

$\|$

$H^i(Y, K_Y + \lceil f^*(K_Y+\Delta)\rceil + f_*L) = 0$ by Kawamata–Viehweg vanishing $\forall i>0$.

VII Log canonical centres

X normal vty, E prime divisor over X, i.e. $f: Y \to X$ birational, Y normal, E a divisor on Y.

Def. $c_X(E) := f(E)$ the centre of E in X.

Def. (X,Δ) log canonical. A log canonical centre W is a centre $W = c_X(E)$ s.t. $a(E,X,\Delta) = -1$.

Then $\mathcal{J}(X,\Delta) \subseteq I_W$ where I_W is the ideal sheaf of W.

W is isolated if $\forall E$ over X, $a(E,X,\Delta) = -1 \Rightarrow c_X(E) = W$.

Lemma. (X,Δ) log canonical, W isolated log canonical centre $\Rightarrow I_W = \mathcal{J}(X,\Delta)$.

Pf: Let $f: Y \to X$ be a log resolution.

$K_Y \sim_{\mathbb{Q}} f^*(K_X+\Delta) + \sum a_i E_i$, $\lceil a_i E_i \rceil = E + F$ where E is reduced, $F \geq 0$ is exceptional.

$0 \to \mathcal{O}_Y(-E+F) \to \mathcal{O}_Y(F) \to \mathcal{O}_E(F|_E) \to 0$

f_*

$0 \to \mathcal{J}(X,\Delta) \to \mathcal{O}_X \to \mathcal{O}_W \to 0 \quad \Rightarrow \mathcal{J}(X,\Delta) = I_W$

Thm. (Shakurov non-vanishing) Let (X,Δ) be a klt pair, L a nef Cartier divisor s.t. $pL-(K_X+\Delta)$ is nef and big for some $p>0$.

Then $|dL|\neq 0$ $\forall d\gg 0$. (This will even turn out to be bpf later.)

Strategy of PROOF:

Modify Δ s.t. (X,Δ) has an isolated lc centre W

$0\to I_W\otimes O_X(dL)\to O_X(dL)\to O_W(dL)\to 0$ by tensoring the ses for W by $O_X(dL)$

Get: $H^0(X,dL)\twoheadrightarrow H^0(W,dL)\to H^1(X,\mathcal{J}(X,\Delta)\otimes mL)=0.$

Thm. (Kawamata subadjunction) (X,Δ) lc pair, W isolated lc centre, A an ample div on X.

Then $(K_X+\Delta+\varepsilon A)|_W \sim_{\mathbb{Q}} K_W+\Delta_W$ $\forall \varepsilon>0$ where (W,Δ_W) is klt.

Ex.
- X smooth, D sm divisor, $(K_X+D)|_D=K_D$
- X quadric cone in $\mathbb{P}^3$ and D a line through the vertex

  $(K_X+D)|_D \sim -\frac{3}{2}H$ where H is a hyperbolic section,

  $\sim K_D+\frac{1}{2}H$ since $K_D\sim -2H$.

Lemma. (Tie breaking) Let (X,Δ) be a klt pair, D a $\mathbb{Q}$-divisor s.t. $(X,\Delta+D)$ is log-canonical, W a minimal lc centre for $(X,\Delta+D)$, A an ample divisor on X.

Then there are arbitrarily small $\varepsilon,\eta>0$, $\varepsilon,\eta\in\mathbb{Q}$ and $D'\sim_{\mathbb{Q}}A$ divisor s.t.

W is an isolated lc centre for $(X,\Delta+(1-\varepsilon)D+\eta D')$.

Pf: Claim. $\exists D'\sim_{\mathbb{Q}}A$ s.t. W is the only lc centre wrt. $(X,\Delta+D)$ contained in Supp D'.

Pf: Take a divisor Z s.t. W is the only lc centre contained in Supp Z.

~~This can be done since W is isolated.~~

Let $D_0\in|mA-Z|$ general for $m\gg 0$. (We need $m\gg 0$ so that $|mA-Z|\neq\emptyset$. This holds due to A being ample.)

Set $D':=\frac{1}{m}(Z+D_0)\sim_{\mathbb{Q}} A$ this is why Supp D' contains W only

$f:Y\to X$ log resolution of $(X,\Delta+D+D')$. (and thus it will be a lr of $(X,\Delta+(1-\varepsilon)D+\eta D')$)

Write $K_Y\sim_{\mathbb{Q}} f^*(K_X+\Delta+D)+\sum_i a_iE_i$, $f^*D=\sum_i b_iE_i$, $K_Y\sim_{\mathbb{Q}} f^*(K_X+\Delta+(1-\varepsilon)D)+\sum_i(a_i+\varepsilon b_i)E_i$

We have $a_i\geq -1$, $a_i+\varepsilon b_i\gneq -1$ for some $\varepsilon_i>0$, then take the minimum of these ε_i to get $\varepsilon>0$

$f^*D'=\sum_i d_iE_i$. If $a_i=-1$ then $d_i>0$ iff $f(E_i)=W$ since W is the only lcc.

$(X,\Delta+(1-\varepsilon)D+\eta D')$ is lc iff $a_i+\varepsilon b_i-\eta b_i\geqq -1$

$\eta:=\min\left\{\frac{\varepsilon b_i}{d_i}\;\middle|\; a_i=-1,\ d_i>0\right\}$, $\varepsilon>0$ any $\Rightarrow$ the pair is lc, W is an isolated lcc

of $(X,\Delta+(1-\varepsilon)D+\eta D')$.

$\forall d_i\geq 0$ since D' is effective since $D'\sim A$ and A is ample and effectiveness is preserved under f

PF OF SHAKUROV THM:

Induction on $n := \dim X$. not really. Instead: $m\cdot(\text{big}) = (\text{ample}) + (\text{effective})$ for some $m \gg 0$, apply for $pL - (K+\Delta)$, $\varepsilon = 1/m$

Kodaira's Lemma: $\exists F \geq 0$, $pL - (K+\Delta) - \varepsilon F$ is ample $\forall\, 0 < \varepsilon < 1$.

Then $(X, \Delta + \varepsilon F)$ klt $\Rightarrow$ wma $pL - (K_X + \Delta)$ is ample.

- $L \equiv 0$: $h^i(X, dL) = h^i(X, K_X + \Delta + \text{ample}) = 0$ by KVV $\forall i > 0$, $d \gg 0$ (a stronger version is needed)

  $\Rightarrow h^0(X, dL) = \chi(X, dL) = \chi(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) = 1$.

  Grothendieck - Riemann - Roch; KVV; these functions are the constants only

  $\underbrace{pL - (K_X + \Delta)}_{\text{ample}} \equiv \underbrace{-K_X + \Delta}_{\text{ample}}$, use Nadel for $K_X + \Delta - K_X - \Delta = \mathcal{O}_X$

  GRR can be avoided by using that χ is a polynomial, $p(d) \geq 0$ $\forall d \geq 0$, $p(0) = 1$.

- $L \not\equiv 0$: Claim. $\forall q \gg 0$ $\exists A$ ample effective $\mathbb{Q}$-divisor, $qL \sim_{\mathbb{Q}} K_X + \Delta + A$ and $(X, \Delta + A)$ not lc.

  $0 < c := \min\{t \in \mathbb{R} \mid (X, \Delta + tA) \text{ not klt}\} \lneq 1$. We assume the Claim for now.

  $(X, \Delta + cA)$ is log canonical. Let W be a minimal lc centre of $(X, \Delta + cA)$.

  Tie breaking: $\exists \varepsilon, \eta$ arbitrarily small s.t. $(X, \underbrace{\Delta + (1-\varepsilon)cA + \eta D'}_{\sim_{\mathbb{Q}} \Delta + c'A \text{ where } 0 < c' < 1})$ is lc where $D' \sim_{\mathbb{Q}} A$.

  Wma W is isolated for $(X, \Delta + cA)$.

  $qL \sim_{\mathbb{Q}} K_X + \Delta + cA + \varepsilon A + (1 - c - \varepsilon) A$

  Subadjunction: $qL|_W \sim_{\mathbb{Q}} K_W + \Delta_W + \underbrace{(1 - c - \varepsilon)}_{>0 \text{ for } \varepsilon \text{ small enough since } 0<c<1} A|_W$

  $qL|_W$ is nef, $qL|_W - (K_W + \Delta_W)$ is ample, (W, Δ_W) is klt

  $\Rightarrow \quad 0 \to I_W(qL) \to \mathcal{O}_X(qL) \to \mathcal{O}_W(qL|_W) \to 0$ $\quad$ W isolated

  $\Rightarrow \quad H^0(X, qL) \twoheadrightarrow \underbrace{H^0(W, qL|_W)}_{\neq 0 \text{ by induction on dimension.}} \to 0$ by Nadel

  It only remains to prove the Claim:

  $$\left(qL - (K_X + \Delta)\right)^n = \left((q-p)L + pL - (K_X + \Delta)\right)^n \geq (q-p)L.\left(pL - (K_X + \Delta)\right)^{n-1} \longrightarrow \infty \text{ as } q \to \infty \text{ because}$$

  $L \not\equiv 0$.

  RR: $h^0(X, d(qL - (K_X + \Delta))) = \chi(X, -)$ by Serre vanishing

  $$= \frac{d^n}{n!}\left(qL - (K_Y + \Delta)\right)^n + \mathcal{O}(n-1).$$

  For $x \in X$ smooth pt, do parameter count: $M_d \in |d(qL - (K_X + \Delta))|$ s.t.

  $\operatorname{mult}_x M_d \geq (n+1)d$. Set $A := \frac{M_d}{d}$

VII. 1. Base point free theorem

Thm. (Bpf thm, Kawamata – Shokurov)

Let (X, Δ) be klt, L nef Cartier divisor, $pL - (K_X + \Delta)$ is nef + big for some $p \in \mathbb{Q}_{>0}$.

Then $|mL|$ is bpf $\forall m \gg 0$.

Lemma. (X, Δ) klt pair, $|D|$ linear system on X, $\dim |D| > 0$ and $S \in |D|$ a general element.

Set $c := \inf\{t \in \mathbb{Q} \mid K_X + \Delta + tS \text{ not klt}\}$ (this is actually a minimum by prev. results). called the lc-threshold

1) If $c < 1$ then any lc centre W of $(X, \Delta + cS)$ is contained in $\mathrm{Bs}\,|D|$

2) If $c = 1$ then one of the following holds:

- $\exists W$ lc centre s.t. $W \subseteq \mathrm{Bs}\,|D|$
- S is reduced and every connected component of S is irreducible, and the lc centres of $(X, \Delta + cS)$ are the components of S.

Pf: Exercise.

$\mu : Y \to X$ log resolution of (X, Δ) s.t. $\mu^* |D| = |L| + \sum r_j F_j$, L is bpf, $\mu(F_j) \subseteq \mathrm{Bs}\,|D|$ whenever $r_j \neq 0$.

PF OF THM: Step 1. Wma $pL - (K_X + \Delta)$ is ample.

Kodaira's Lemma: $\exists F \geq 0$ s.t. $pL - (K_X + \Delta) - \varepsilon F$ is ample for $0 < \varepsilon \ll 1$

Now take $(X, \Delta + \varepsilon F)$, this is klt for $0 < \varepsilon \ll 1$.

Step 2. For any $d \geq 2$ $\exists e \in \mathbb{N}_{>0}$ s.t. $\mathrm{Bs}\,|d^e L| = \emptyset$.

$\lightning$ $\exists d \geq 2$ $\forall e \in \mathbb{N}_{>0}$: $\mathrm{Bs}\,|d^e L| \neq \emptyset$

$\mathrm{Bs}\,|d^{m+1} L| \subseteq \mathrm{Bs}\,|d^m L|$ $\Rightarrow$ $\mathrm{Bs}\,|d^e L|$ stabilises for $e \gg 0$ by noetherian induction (these are closed sets)

Up to taking a multiple of L wma

$\mathrm{Bs}\,|dL| = \mathrm{Bs}\,|d^e L|$ $\forall e \geq 1$. and $|dL| \neq 0$ (nonvanishing)

Take $D \in |dL|$ general and set $c := \{t \in \mathbb{Q} \mid K_X + \Delta + tD \text{ is not klt}\}$

Take an lc centre W of $(X, \Delta + cD)$ s.t. $W \cap \mathrm{Bs}\,|dL| \neq \emptyset$.

Take $q = d^e \geq p + d$ and write $qL \sim_{\mathbb{Q}} K_X + \Delta + D + A$ where A is ample. $\mathbb{Q}$-Cartier.

Tie breaking: $\exists \varepsilon, \eta$ small enough s.t. W is an isolated lc centre of $(X, \Delta + (1-\varepsilon)cD + \eta A)$.

$\Rightarrow I_W = \mathcal{J}(X, \Delta + (1-\varepsilon)cD + \eta A)$

Write $qL \sim_{\mathbb{Q}} K_X + \Delta + (1-\varepsilon)cD + \eta A + \underbrace{(1-\eta)A + \varepsilon c D + (1-c)D}_{\text{ample b/c } A \text{ ample, } D \text{ nef}}$

Kawamata subadjunction $\Rightarrow$ $qL|_W = K_W + \Delta_W + A_W$ where A_W is ample

(W, Δ_W) is klt, $qL|_W - (K_W - \Delta_W)$ is ample $\Rightarrow$ $|mL|$ is bpf for $m \gg 0$. by induction

Nadel vanishing: $H^0(X, qL) \twoheadrightarrow H^0(W, mL|_W)$ $m \gg 0$ $\lightning$

Cor. • (X,Δ) klt, $K_X+\Delta$ nef + big $\Rightarrow$ $K_X+\Delta$ is semiample (has a bpf multiple).

• X terminal, K_X nef + big $\Rightarrow$ K_X semiample, $\varphi_{|m K_X|}: X \to X^{can} := \mathrm{Proj} \bigoplus_{d\geq 1} H^0(X, dK_X)$

X^{can} has canonical singularities and $K_{X^{can}}$ is ample.

Conj. (Abundance) • (X,Δ) lc, $K_X+\Delta$ nef $\Rightarrow$ $K_X+\Delta$ is semiample.

• X terminal, K_X nef $\Rightarrow$ K_X semiample.

Rmk. Known for $\dim \leq 3$, but it's not trivial even for surfaces.

Rmk. Bpf thm. false for (X,Δ) lc but true if $pL-(K_X+\Delta)$ is ample, the proof is similar to the one we gave.

Ex. (Zariski) C smooth cubic in $\mathbb{P}^2$, l line in $\mathbb{P}^2$. Take $p_1,\dots,p_{12}\in C$ points s.t. $\mathcal{O}_C(l+C-(p_1+\dots+p_{12}))$ is not torsion.

$X := Bl_{p_1,\dots,p_{12}}\mathbb{P}^2 \xrightarrow{f} \mathbb{P}^2$, $\Delta := C'$ the strict transform of C

$$L := f^*\mathcal{O}_{\mathbb{P}^2}(1)+C' = 4f^*\mathcal{O}(1)-\sum_{i=1}^{12}E_i$$

(X,Δ) lc, L nef. $L-(K_X+\Delta)$ nef + big

But $C'\subseteq Bs|mL|$ $\forall m\geq 1$. $\longrightarrow$ counterexample of Bpf Thm. for lc

$R(X,L)$ (the canonical ring) is not finitely generated.

VII.2. Rationality theorem.

Thm. (Rationality, Kawamata)

Let (X,Δ) be klt, $(K_X+\Delta)$ not nef, $a\in\mathbb{N}_{>0}$ s.t. $a(K_X+\Delta)$ is Cartier, H a nef + big Cartier divisor.

Set $r := r(H) := \sup\{t\in\mathbb{Q} \mid H+t(K_X+\Delta) \text{ is nef}\}$ nef threshold

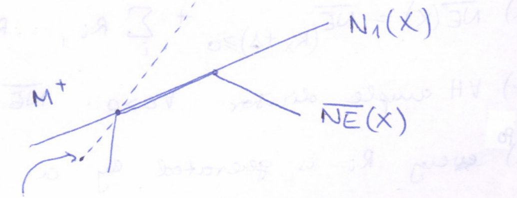
Then $r\in\mathbb{Q}$, and $r=\frac{u}{v}$, $u,v\in\mathbb{Z}$, $\gcd(u,v)=1$ and $v\leq a(\dim X+1)$.

How do we apply this? $\longrightarrow$ we will use it for the Cone Thm. in lec. 11.

X smooth, $\Delta=0$, H ample, K_X not nef

$M = H+rK_X$ is nef and $\mathbb{Q}$-Cartier

$\frac{1}{r}M-K_X=\frac{1}{r}H$ is ample $\xRightarrow{\text{Bpf Thm.}}$ dM bpf for $d\gg 0$



we will need the bound on the denominator for this perturbation

Lemma. $P(x,y)$ polynomial (nontrivial), $\deg P \leq n$. Fix $a \in \mathbb{N}_{>0}$, $\varepsilon \in \mathbb{R}_{>0}$.

Assume $\exists r \in \mathbb{R}$ s.t. P vanishes for all sufficiently large $x,y \in \mathbb{N}$. s.t. $0 < ay - rx < \varepsilon$

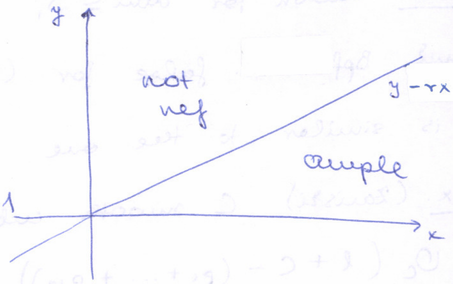
Then $r \in \mathbb{Q}$, and if $r = \frac{u}{v}$ then $v \leq \frac{a(n+1)}{\varepsilon}$.

PF OF THM (SKETCH):

Assume X is smooth, $\Delta = 0$, H is ample. (Somehow this isn't that big of a reduction.)

$\nexists$ $r \notin \mathbb{Q}$. $\Rightarrow$ $\exists$ infinitely many pairs $(p,q) \in \mathbb{N}$ s.t. $0 < q - rp < 1$. $\left(\Leftrightarrow \frac{q-1}{p} < r < \frac{q}{p}\right)$

$P(x,y) := \chi(X, xH + yK_X)$



For (p,q) as above, $pH + qK_X - K_X$ is ample (we are close to the line)

Lemma $\Rightarrow$ $h^0(X, pH + qK_X) \neq 0$ for inf. many (p,q) s.t. $0 < q - rp < 1$

$S \in H^0(pH + qK_X)$ s.t. $c := \inf\{t \in \mathbb{Q} \mid K_X + \Delta + tS \text{ not klt}\} < 1$

Tie-breaking, Nadel vanishing $\Rightarrow$ $H^0(X, pH + qK_X) \twoheadrightarrow H^0(W, (pH + qK_X)|_W)$

$c < 1 \Rightarrow W \subseteq \mathrm{Bs}\,|pH + qK_X|$

Repeat for $P_W(x,y) := h^0(W, xH + yK_X) \xrightarrow{\text{Lemma}} \neq 0$ for inf many $(p',q') \in \mathbb{N}^2$ $\lightning$ $\square$

PF OF LEMMA:

$\nexists$ $r \notin \mathbb{Q}$. Then $\exists$ infinitely many $(p,q) \in \mathbb{N}^2$ s.t. $0 < |aq - rp| < \frac{\varepsilon}{n+2}$.

So there is (p_0, q_0) in that range s.t. $P(p_0, q_0) = 0$.

Also $(2p_0, 2q_0), \ldots, ((n+1)p_0, (n+1)q_0)$ are zeros of P since they lie in the range.

$\Rightarrow q_0 x - p_0 y \mid P(x,y)$

Repeat the argument for different (p_i, q_i) pairs, $n+1$ of them

$\rightarrow n \geq \deg P \geq n+1$ $\lightning$

A similar argument works for the second part. $\square$

VII.3. Cone and contraction theorems

23.01.2019

Thm. (Cone) Let (X,Δ) be a projective klt pair. Then there are countably many $(K_X + \Delta)$-negative extremal rays $R_i \subseteq \overline{NE}(X)$ s.t.

a) $\overline{NE}(X) = \overline{NE}_{(K_X+\Delta)\geq 0} + \sum_i R_i$, $R_i = \overline{NE} \cap L_i^{\perp}$ where L_i is a nef Cartier divisor

b) $\forall H$ ample divisor $\forall \varepsilon > 0$: $\overline{NE}(X) = \overline{NE}_{(K_X+\Delta+\varepsilon H)\geq 0} + \sum_{\text{fin}} R_i$

c) every R_i is generated by a rational curve (PO, uses Mori's bend and break)

Notation. $\overline{NE}_{(K_X+\Delta)\geq 0} := \overline{NE} \cap \{\text{part where } K_X + \Delta \geq 0\}$

Ex. • X smooth Fano ($-K_X$ ample) $\Rightarrow$ $\overline{NE}(X) = \sum_{\text{fin}} R_i$

• X of general type (K_X big), $K_X \equiv A + E$, A ample, E effective

Take $\varepsilon \in \mathbb{Q}$, $0 < \varepsilon < 1$ s.t. $(X, \varepsilon E)$ is Klt.

$\overline{NE}(X) = \overline{NE}(X)_{(K_X + \varepsilon E + \varepsilon A) \geq 0} + \sum_{\text{fin}} R_i$ $\Rightarrow$ there are only fin many K_X-negative extremal rays

$(1+\varepsilon) K_X \geq 0$

• X abelian vty $\Rightarrow$ $\overline{NE}(X)$ is round

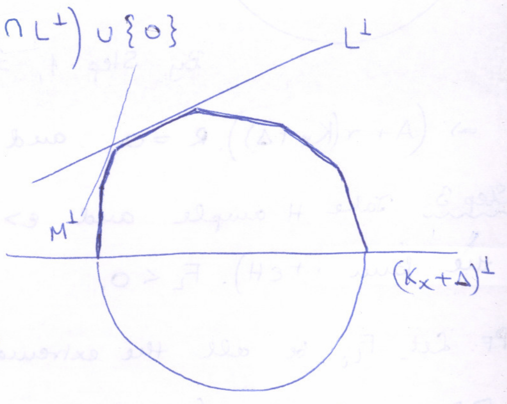
• X blowup of $\mathbb{P}^2$ at more than 9 general pts $\Rightarrow$ there are inf many (-1)-curves $\Rightarrow$ inf many extremal rays.

PF OF THM: Wma $K_X + \Delta$ is not nef.

Step 1. Let L be a nef Cartier divisor s.t. $\overline{NE}_{K_X + \Delta < 0} \cap L^{\perp} \neq \emptyset$

$\Rightarrow$ $\exists M$ nef divisor and R extremal ray, $R \subseteq (\overline{NE}_{K_X + \Delta < 0} \cap L^{\perp}) \cup \{0\}$

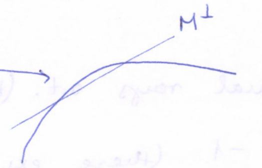
s.t. $R = \overline{NE}(X) \cap M^{\perp}$



Rmk. $F_L := \overline{NE}(X) \cap L^{\perp}$ is an extremal face of $\overline{NE}(X)$.

$v_1, v_2 \in \overline{NE}(X) \Rightarrow v_1 + v_2 \in F_L \Rightarrow v_1, v_2 \in F_L$ b/c $L.v_1 = 0$

this cannot happen



PF OF STEP 1: Let H be an ample Cartier divisor. For any $d \in \mathbb{N}$ let

$r_L(d, H) := \sup \{ t \in \mathbb{Q} \mid dL + H + t(K_X + \Delta) \text{ is nef} \} \in \mathbb{Q}$ nef threshold.

L is nef $\Rightarrow$ $r_L(d, H)$ is a non-decreasing fuction in d

Take $z \in \overline{NE}_{K_X + \Delta < 0} \cap L^{\perp}$. Then $(dL + H + r_L(d,H) \cdot (K_X + \Delta)) z \geq 0$. $\Leftrightarrow$ $r_L(d,H) \leq \dfrac{H.z}{-(K_X+\Delta).z}$

Notice that this bound is uniform, i.e. independent of d.

The denominator of $r_L(d,H)$ is bounded by $a(\dim X + 1)$ where $a \in \mathbb{N}$ s.t. $a(K_X + \Delta)$ is Cartier. $\Rightarrow$ $r_L(d,H)$ is constant for $d \geq d_0$ for some $d_0 \in \mathbb{N}$

Set $M(dL, H) := dL + H + r_L(d,H)(K_X + \Delta)$ nef. not ample

For $d > d_0$: $\{0\} \neq F_{M(dL,H)} \subseteq F_L$ (this is why we are bothering with d and don't just take $d=1$)

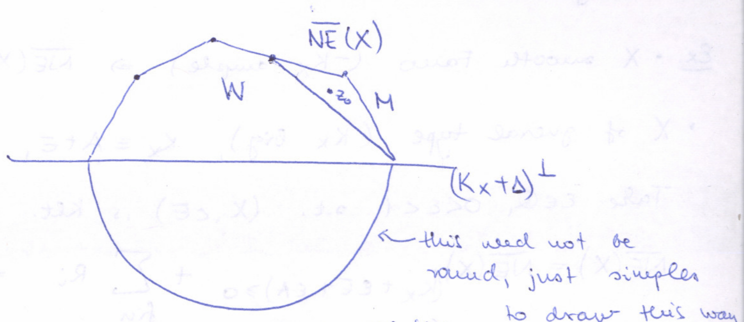
and $F_{M(dL,H)} \subseteq \overline{NE}_{(K_X+\Delta)<0} \cup \{0\}$

If $\dim F_{M(dL,H)} = 1$ then we are done. ✓

If $\dim F_{M(dL,H)} \geq 2$ then take $H_1, \dots, H_g$ be a basis of $N^1(X)$ s.t. $\forall H_i$ ample

$\Rightarrow (dL + H_i + r_L(d, H_i)(K_X + \Delta))|_{F_{M(dL,H)}} \neq 0$ for some i $\Rightarrow$ we really cut down the face $F_{M(dL,H)}$

Step 2: $\overline{NE}(X) = \overline{NE}(X)_{K_X+\Delta\geq 0} + \overline{\sum_{\substack{L \text{ nef} \\ \dim F_L = 1}} F_L} =: W$



Pf.:

Clear: $\overline{NE}(X) \supseteq W$. Assume $\overline{NE}(X) \supsetneq W$.

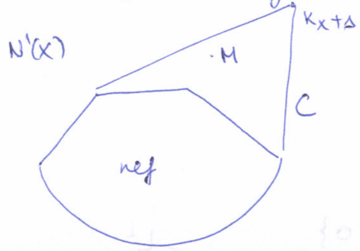
$\Rightarrow \exists z_0 \in \overline{NE}(X)\setminus W$ and a Cartier divisor M

s.t. $M.z_0 < 0$, $M.(W\setminus\{0\}) > 0$

We will do perturbation by some ample divisor and obtain a contradiction.

Let C be the dual cone of $\overline{NE}_{(K_X+\Delta)\geq 0}$: $C := \{ D \in N^1(X) \mid D.z \geq 0 \ \forall z \in \overline{NE}_{(K_X+\Delta)\geq 0}\}$

C is generated by $K_X+\Delta$ and nef divisors, M is in the interior of C



$\Rightarrow M = A + p(K_X+\Delta)$ where A is ample, $p \in \mathbb{Q}_{>0}$

Rationality thm $\Rightarrow r := \sup\{ t \in \mathbb{Q} \mid A + t(K_X+\Delta) \text{ nef}\} \in \mathbb{Q}$,

$D = A + r(K_X+\Delta)$ is nef s.t. $D^\perp \cap \overline{NE}_{K_X+\Delta<0} \neq \emptyset$

By Step 1, $\exists R$ extremal ray, $R \subseteq F_D$ (face given by D)

$\Rightarrow (A + r(K_X+\Delta)).R = 0$ and $(A + p(K_X+\Delta)) > 0 \Rightarrow p < r \Rightarrow M$ is ample $\lightning$ $\boxed{2}$

Step 3: Take H ample and $\varepsilon > 0$. There are only finitely many extremal rays F_L s.t. $(K_X+\Delta+\varepsilon H).F_L < 0$.

Pf.: Let F_{L_i} be all the extremal rays s.t. $(K_X+\Delta+\varepsilon H).F_{L_i} < 0$.

Take $z_i \in F_{L_i}$ s.t. $(K_X+\Delta).z_i = -1$ (these exist) $\Rightarrow 0 < H.z_i < \varepsilon \ \forall i$ $(*)$

Let $D_1, \dots, D_\varrho$ be a basis of $N^1(X)$ consisting of ample Cartier divisors

Serre $\Rightarrow \exists m > 0$: $mH - D_i$ is ample $\forall i$

$(mH - D_i)z_j > 0$ by ampleness $\Rightarrow D_i.z_j < mH.z_j \overset{(*)}{<} m\varepsilon$

As in Step 1: for any R_{L_j} and D_i we can find a nef divisor $\left(dL_j + D_i + r_{L_j}(d, D_i)(K_X+\Delta)\right)z_j > 0$

$\Rightarrow D_i.z_j = r_{L_j}(d, D_i)$ has bounded denominator by rat. thm.

$\Rightarrow$ In the dual basis of $D_1, \dots, D_\varrho$, the coordinates of z_j have only finitely many possible values $\Rightarrow$ there are only finitely many. $\boxed{3}$

Steps 2 & 3 imply the assertion of the Thm: $\overline{NE} = \overline{NE}_{(K_X+\Delta+\varepsilon H)\geq 0} + \sum_{\text{fin}} R_i \Rightarrow$ b);

the accumulation pts of $\sum R_i$ are on $K_X+\Delta = 0 \Rightarrow$ a).

Thm. (Contraction) Let (X,Δ) be a projective Klt pair and R a $(K_X+\Delta)$-negative extremal ray. Then there is a unique morphism $\varphi: X \to Z$ where Z is a projective normal variety s.t. 1) $\varphi_* \mathcal{O}_X = \mathcal{O}_Z$ and $\varphi(C) = pt$ for a curve $C \subseteq X$ iff $[C] \in R$
2) if D is a Cartier divisor on X s.t. $D.R = 0$ then $D = \varphi^* D_Z$ for some D_Z Cartier.

Pf: By the Cone Thm. (or its proof) there is a nef divisor L s.t. $R = F_L$ is the face associated to L. ← part b), appropriate choice of H and ε

Kleiman's Criterion: $mL - (K_X+\Delta)$ is ample $\forall m \gg 0$

Bpf Thm: $|mL|$ is bpf $\forall m \gg 0$, $\varphi := \varphi_{|mL|}$

This proves 1). Use Stein factorisation to see that φ has conn fibres. See [Lazarsfeld].

For $m \gg 0$: $mL = \varphi^*(H_1)$, $(m+1)L = \varphi^*(H_2) \overset{\text{substract}}{\Rightarrow} L = \varphi^*(H)$ where H, H_1, H_2 are Cartier on Z.
$H = H_2 - H_1$

For D Cartier on X s.t. $D.R = 0$: for $m \gg 0$, $mL + D$ is nef and $R = \overline{NE} \cap (mL+D)^\perp$
$\Rightarrow mL + D = \varphi^*(D_Z)$ as before.

Cor. (X,Δ) projective Klt pair, R $(K_X+\Delta)$-negative extremal ray with $[C] \in R$, contraction $\varphi: X \to Z$.
Then there is an exact sequence $0 \to \operatorname{Pic} Z \xrightarrow{\varphi^*} \operatorname{Pic} X \longrightarrow \mathbb{Z}$

$$L \longmapsto L.C$$

In particular $\rho(Z) = \rho(X) - 1$ holds for the Picard numbers.

Pf: φ^* is injective: $\varphi_* \varphi^* L = \varphi_* \mathcal{O}_X \otimes L = L$

Exactness at $\operatorname{Pic} X$ comes from b) in the Cone Thm.

Def / Lemma 1. X normal proj vty and $\mathbb{Q}$-factorial. Let $f: X \to Y$ be a contraction of an extremal ray $R \subseteq \overline{NE}(X)$. Then one of the following holds: 30.01.20

(1) (Fibre type) $\dim Y < \dim X$
(2) (Divisorial) $\operatorname{Exc}(f)$ is an irred divisor and f is birational
(3) (Small) $\operatorname{codim} \operatorname{Exc}(f) \geq 2$ and f is birational

Pf: (2) Let $\operatorname{Exc}(f) \supseteq E$ be an irred. component. Take $p \in \operatorname{Exc}(f)$. Fibres connected $\Rightarrow$
$\exists C$ curve, $p \in C \subseteq \operatorname{Exc}(f)$ s.t. $f(C) = pt$, $[C] \in R$

Now E is $\mathbb{Q}$-Cartier + exceptional $\Rightarrow E.R < 0$ by Neg. Lemma. $\Rightarrow E.C < 0$
$\Rightarrow p \in C \subseteq E$.

Lemma 2. Let X be a proj vty with terminal (or Klt) singularities, $f: X \to Y$ the contraction of a K_X-negative extremal ray. Assume X to be $\mathbb{Q}$-factorial, and f divisorial or of fibre type. Then Y is $\mathbb{Q}$-factorial.

PF: Assume f to be divisorial, the other case is similar.

Let B be a Weil divisor on Y. Then $f_*^{-1}B$ is $\mathbb{Q}$-Cartier on X, so $E := \mathrm{Exc}(f)$ is a prime $\mathbb{Q}$-Cartier divisor. $\exists r \in \mathbb{Q}: (f_*^{-1}B + rE).R = 0$ since $E.R < 0$ by Neg. Lemma.

Cone thm $\Rightarrow$ $f_*^{-1}B + rE \underset{\mathbb{Q}}{\sim} f^*D$ for D $\mathbb{Q}$-Cartier on Y.

$\overset{f_*}{\Rightarrow}$ $B \underset{\mathbb{Q}}{\sim} D$ is $\mathbb{Q}$-Cartier $\to$ Y is $\mathbb{Q}$-factorial. $\square$

Rmk. $f: X \to Y$ small contraction of a K_X-negative extremal ray. (X Klt)

Then K_Y is <u>not</u> $\mathbb{Q}$-Cartier, {otherwise $K_X = f^*K_Y$ but $0 > R.K_X = R.f^*K_Y = 0$ ↯}

We can't even do intersections with K_Y. Solution: soon.

Prop. Consider the cd

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

where X, X', Y proper normal vties, f, f' proper normal morph.

Assume that (1) $-K_X$ is $\mathbb{Q}$-Cartier and f is nef
(2) $K_{X'}$ is $\mathbb{Q}$-Cartier and f' is nef

Then for any exc. divisor E over Y: $a(E, X) \leq a(E, X')$

PF: Let E be an exc div over Y. $\Rightarrow \exists Z \subseteq E$ normal variety and cd:

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[dr, "g'"] \arrow[dd, "h"] & \\ X \arrow[rr, dashed] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

$$K_Z - \sum a_i(E_i, X) E_i \sim g^* K_X$$

$$K_Z - \sum a_i(E_i, X') E_i \sim (g')^* K_{X'}$$

$$\Rightarrow \left(\sum a_i(E_i, X) - \sum a_i(E_i, X')\right) E_i \sim (g')^* K_{X'} - g^* K_X \text{ is an}$$

h-exceptional and h-nef divisor (do checking by case work: distinguish b/w cases based on whether f, f', g or g' contracts sth.)

$\to a_i(E_i, X) \leq a_i(E_i, X')$ (Somewhere we used Neg. lemma again.) $\square$

Cor. X normal proj $\mathbb{Q}$-factorial, $f: X \to Y$ divisorial contraction of a K_X-negative extremal ray. If X is terminal (resp. canonical resp. Klt) then Y is terminal (resp. can. resp. Klt).

PF: Lemma 2 $\Rightarrow$ Y is $\mathbb{Q}$-factorial

Apply Prop with $X' = Y$, the vertical morphism is id. $\square$

Rmk. There is a K_X-negative divisorial contraction of an extremal ray s.t. X is terminal but Y is <u>not</u> (K_Y is not $\mathbb{Q}$-Cartier). This is why we assume $\mathbb{Q}$-factoriality in the beginning.

Mori theory, lecture 12

Rmk. It is possible to give conditions under which the inequality is strict.

Def. (Flip) Let X be a normal vty, K_X $\mathbb{Q}$-Cartier, $f : X \to Y$ small contraction of a K_X-negative extremal ray. A flip of f is a cd

$$\begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$$

such that (1) K_{X^+} is $\mathbb{Q}$-Cartier

(2) f^+ is the contraction of a K_{X^+}-positive extremal ray

(3) $\operatorname{codim} \operatorname{Exc}(f^+) \geq 2$.

This is analogous to surgery in topology. Remember that our goal is still to get rid of negative curves. A similar notion is that of flops.

Prop. Let $\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$ be a flip. Then

1) X terminal $\Rightarrow$ X^+ terminal

2) X $\mathbb{Q}$-factorial $\Rightarrow$ X^+ $\mathbb{Q}$-factorial

3) Given $f : X \to Y$ then if the flip exists then it is unique, and

$$X^+ = \operatorname{Proj}_Y \bigoplus_{m \geq 0} f_* \mathcal{O}_Y (m K_X)$$

Pf: 1) From prev Prop.

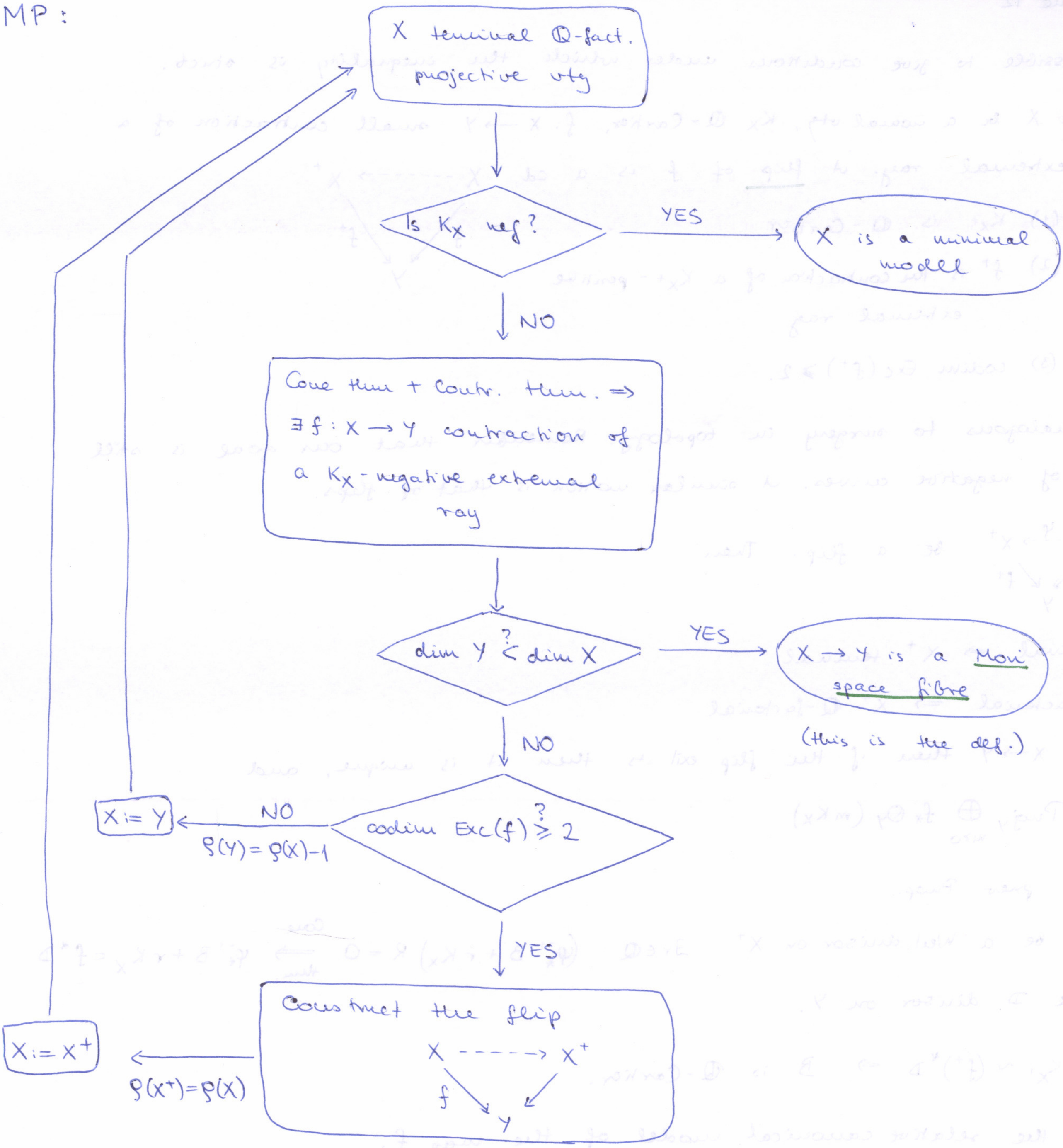
2) Let B be a Weil divisor on X^+. $\exists r \in \mathbb{Q} : (\varphi_*^{-1} B + r K_X) R = 0 \overset{\text{Cone}}{\underset{\text{thm.}}{\Longrightarrow}} \varphi_*^{-1} B + r K_X = f^* D$ for some D divisor on Y.

$\overset{\varphi_*}{\Longrightarrow} B + r K_{X^+} \sim (f^+)^* D \Rightarrow B$ is $\mathbb{Q}$-Cartier.

3) X^+ is the relative canonical model of the map f.

Then one has to verify that it satisfies all the conditions of a flip.

MMP :



<u>Questions</u>:
- Can we always construct the flip?
- Does this process always terminate?

<u>Mori (1988)</u>: In dimension 3, flips exist (hard) and the process terminates (easy).

<u>Thm.</u> (KMM, 1990) The MMP works in dimension 4. (Uses difficulty + cycles in codim 2).

<u>Thm.</u> (BCHM, 2010) Flips exist in the Klt case.

<u>Thm.</u> X smooth vty $\Rightarrow \bigoplus_{m \geq 0} H^0(X, mK_X)$ is finitely generated.

<u>Thm.</u> X of general type (i.e. K_X is big) $\Rightarrow \exists$ MMP which terminates with a minimal model.

<u>Def.</u> <u>(Difficulty)</u> X terminal, <u>$d(X)$</u> $:= \#\{E \mid E$ exceptional divisor over X s.t. $a(E, X) < 1\} \in \mathbb{Z}$